Liquidity Provision in Concentrated Liquidity Markets

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Automated Market Makers

Constant Function Market Makers

- A pool with assets X and Y
- Available liquidity x and y
- Deterministic trading function $f(x, y)$

  $\Longrightarrow$ defines the state of the pool before and after a trade
- Liquidity providers (LPs) deposit assets in the pool.

  Liquidity takers (LTs) trade with the pool.

Liquidity providers in a CFMM

LP trading condition

LP trading condition

- LPs change the depth:

$$f\left(x+\Delta x, y+\Delta y\right)=\overline{\kappa}^{2}>f\left(x, y\right)=\kappa^{2}.$$

- LP trading condition: LP operations do not change the rate:

$$Z=-\varphi^{\kappa\prime}(y)=-\varphi^{\overline{\kappa}\prime}(y+\Delta y)$$

- LPs hold a portion of the pool and earn fees.

LP trading condition

In CPMMs

- LP trading condition:

$$\frac{x + \Delta x}{y + \Delta y} = \frac{x}{y}$$

- Depth variations

$$\overline{\kappa}^2 = (x + \Delta x)(y + \Delta y) > \kappa = x\, y$$

LP trading condition

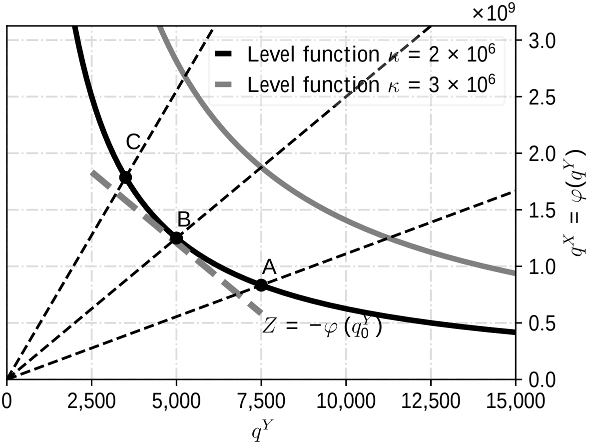
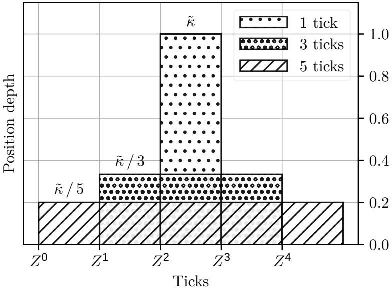


Figure: Geometry of CPMMs: level function $\varphi\left(q^Y\right) = q^X$ for two values of the pool depth.

Concentrated liquidity

Concentrated liquidity: definition

- Price is discretised in Ticks: $\{Z^1, \ldots, Z^N\}$.
- Two consecutive ticks $[Z^i, Z^{i+1}]$: tick range.
- LPs can post liquidity with depth $\tilde{\kappa}^{\ell,u}$ between two ticks $(Z^\ell, Z^u]$.



Concentrated liquidity: geometry

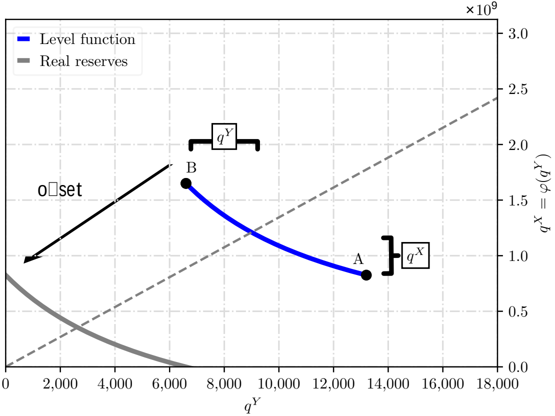


Figure: Geometry of CPMMs with CL. Key formula for an LP providing liquidity in the range $[Z^A, Z^B]$: $\left(q^X + \tilde{\kappa}\sqrt{Z^A}\right)\left(q^Y + \tilde{\kappa}\frac{1}{\sqrt{Z^B}}\right) = \tilde{\kappa}^2$

Concentrated liquidity: geometry

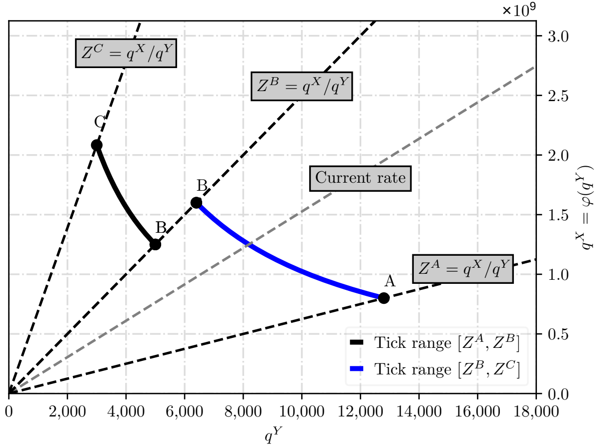


Figure: Geometry of CPMMs with CL: two adjacent tick ranges $[Z^B, Z^C]$ and $[Z^A, Z^B]$ with different liquidity depth.

Concentrated Liquidity: effects of concentration

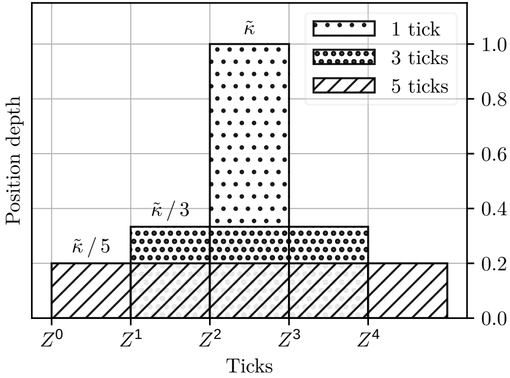


Figure: Position depth for three LP ranges. The first is concentrated over a range of one tick, the second over a range of three ticks, and the last over a range of five ticks.

Concentrated Liquidity: what it looks like

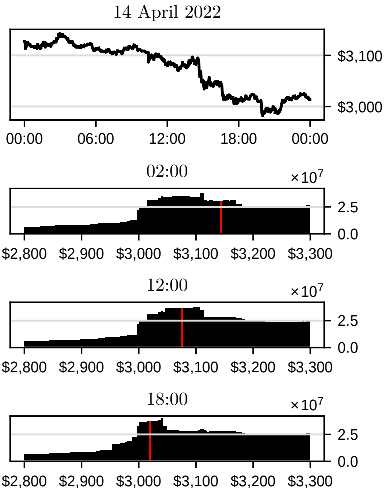
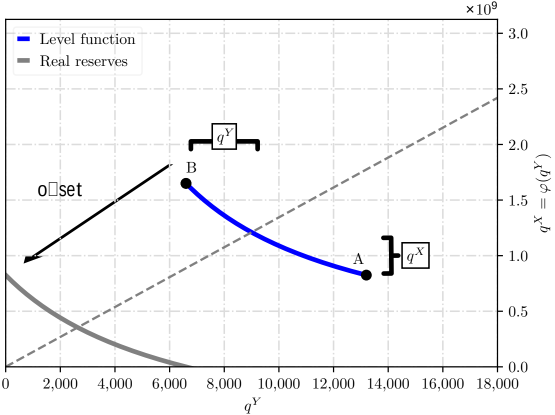


Figure: ETH/USDC rates on 14 April 2022.

Contributions & results

LP wealth: position value

LP wealth: position value



LP wealth: position value

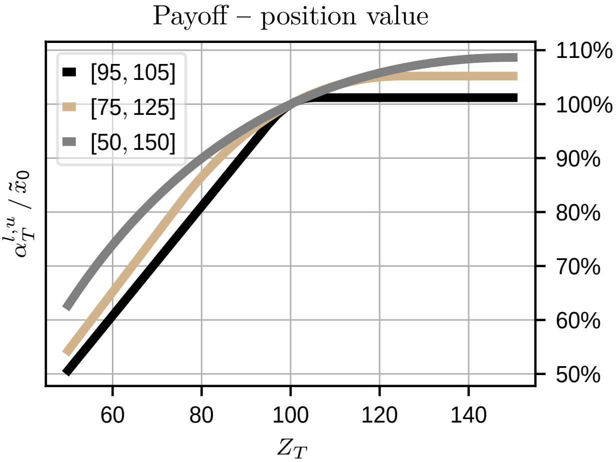


Figure: Terminal value of the LP's assets as a Payoff ≈ short put option.

LP wealth: position value

Setup:
Liquidity range: $[Z_0 - \delta, Z_0 + \delta]$.
Market : $\mathbf{Z_0} = 100$, **vol**=2%, 5%, 10%, **drift**=0%
$\mathbf{T} = 1$ day, **Pool size** = $\$200 \times 10^6$.

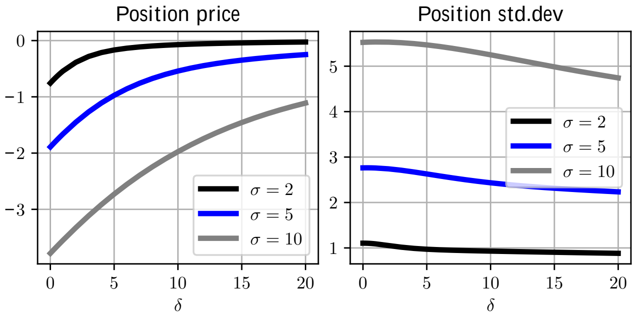


Figure: Price and risk of the LP's option.

LP wealth: position value

- Dynamic strategy: target the rate $(Z^{\ell}, Z^{u}] \ni Z$.

- LP wealth dynamics $\tilde{x}$ in discrete-time:

$$\tilde{x}_{t+\Delta t} - \tilde{x}_t = 2\,\tilde{x}_t \left(\frac{1}{\delta_t^{\ell} + \delta_t^{u}}\right) \left(2\,\frac{Z_{t+\Delta t}^{1/2} - Z_t^{1/2}}{Z_t^{1/2}} - \frac{Z_{t+\Delta t} - Z_t}{Z_t}\left(1 - \frac{\delta_t^{u}}{2}\right)\right),$$

where

$$Z^{u} = Z / \left(1 - \delta^{u}/2\right)^2 \quad \text{and} \quad Z^{\ell} = Z \times \left(1 - \delta^{\ell}/2\right)^2 .$$

- For small values of $Z^{u} - Z^{\ell}$:

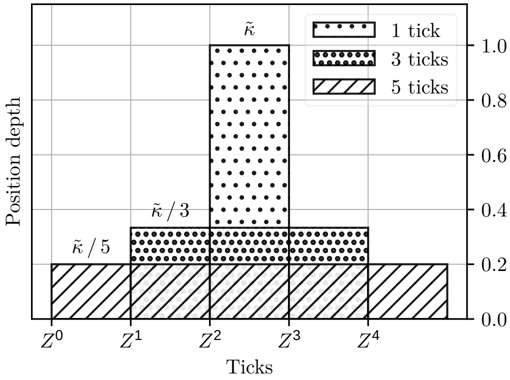
$$\left(Z^{u} - Z^{\ell}\right) / Z = \left(1 - \delta^{u}/2\right)^{-2} - \left(1 - \delta^{\ell}/2\right)^{2} \approx \delta^{u} + \delta^{\ell} .$$

- In continuous-time. If $dZ_t = \mu_t\, Z_t\, dt + \sigma\, Z_t\, dW_t$, then

$$\boxed{d\tilde{x}_t = \tilde{x}_t \left(\frac{1}{\delta_t^{\ell} + \delta_t^{u}}\right) \left(-\frac{1}{4}\sigma^2 dt + \mu_t\, \delta_t^{u}\, dt + \sigma\, \delta_t^{u} dW_t\right)}$$

LP wealth: fees

LP wealth: premium (fees)



Wealth dynamics for dynamic LPs

- **Assumption 1**: The pool generates fees at a stochastic rate π.

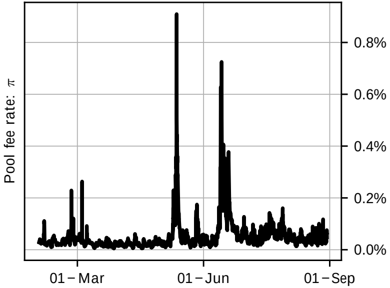


Figure: Pool fee rate from February to August 2022 in ETH/USDC pool.

- **Fee revenue**: $$dp_t = \underbrace{(\tilde{\kappa}_t \,/\, \kappa)}_{\text{Position depth}} \underbrace{\pi_t}_{\text{Fee rate}} \underbrace{2\,\kappa\, Z_t^{1/2}}_{\text{Pool size}} \, dt = \left(\frac{4}{\delta_t^\ell + \delta_t^u}\right) \pi_t\, \tilde{x}_t \, dt \,.$$
- Problem in continuous-time: $$\tilde{\kappa}_t = 2\, \tilde{x}_t \left(\frac{1}{\delta_t^\ell + \delta_t^u}\right) Z_t^{-1/2} \,.$$

Wealth dynamics for dynamic LPs

- **Assumption 2**: Concentration cost is quadratic in the spread.
- **Fee revenue**: $$\mathrm{d}p_t = \left(\frac{4}{\delta^\ell_t + \delta^u_t}\right) \pi_t\, \tilde{x}_t\, \mathrm{d}t \; - \gamma \left(\frac{1}{\delta^\ell_t + \delta^u_t}\right)^2 \tilde{x}_t\, \mathrm{d}t\,.$$

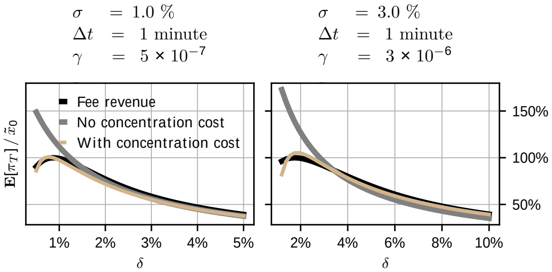


Figure: Fee income without concentration cost and with concentration cost using simulations of Z and π.

Optimal LP strategy

Closed-form optimal positions

Wealth dynamics for dynamic LPs

- Wealth dynamics

$$\mathrm{d}\tilde{x}_t = \frac{1}{\delta_t}\left(4\,\pi_t - \frac{\sigma^2}{2}\right)\tilde{x}_t\,\mathrm{d}t + \mu_t\,\rho\left(\delta_t, \mu_t\right)\tilde{x}_t\,\mathrm{d}t + \sigma\,\rho\left(\delta_t, \mu_t\right)\tilde{x}_t\,\mathrm{d}W_t - \frac{\gamma}{\delta_t^2}\,\tilde{x}_t\,\mathrm{d}t\,.$$

- Performance criterion $u^\delta(t, \tilde{x}, z, \pi, \mu) = \mathbb{E}_{t,\tilde{x},z,\pi,\mu}\left[U\left(\tilde{x}_T^\delta\right)\right]$.

- Optimal strategy for log-utility:

$$\boxed{\delta_t^\star = \frac{2\,\gamma + 2\,\sigma^2\mu^2}{\Pi_t + \mu^2 - \sigma^2\,\left(\mu + \frac{1}{4}\right)}}$$

- When $\mu = 0$,

$$\boxed{\delta_t^\star = \frac{2\,\gamma}{\Pi_t - \frac{\sigma^2}{4}}}$$

Optimal width as a function of profitability

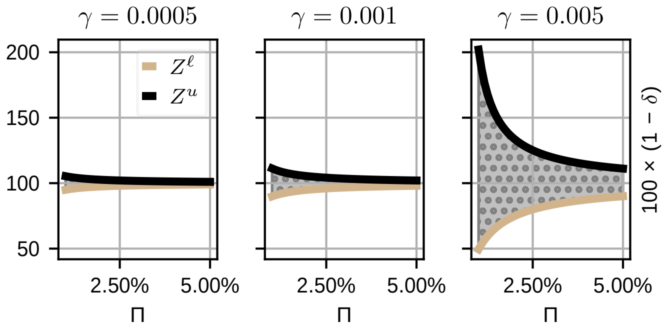


Figure: Optimal LP position range $(Z^\ell, Z^u]$ as a function of the pool fee rate Π for different values of the cost parameter γ, when $Z = 100$, $\sigma = 0.02$, and $\mu = 0$.

Optimal width as a function of PL

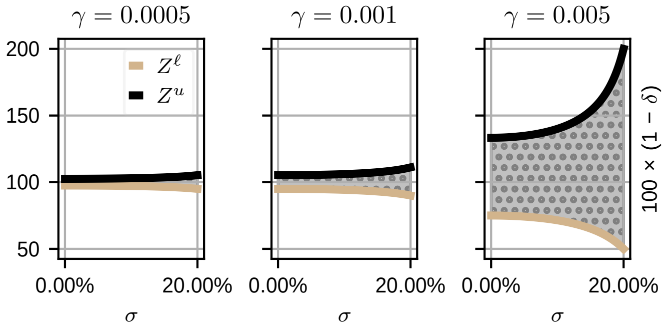


Figure: Optimal LP position range $(Z^\ell, Z^u]$ as a function of the volatility σ for different values of the cost parameter γ, when $Z = 100$, $\Pi = 0.02$, and $\mu = 0$.

Optimal width as a function of the trend

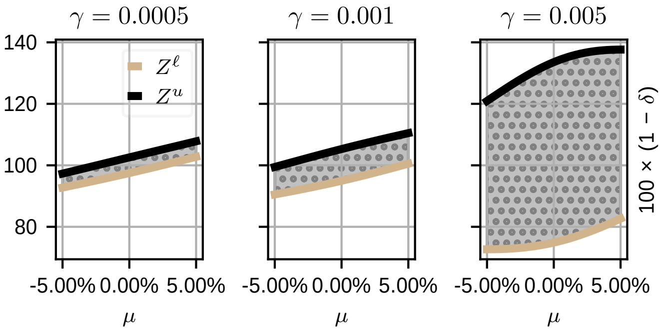


Figure: Optimal LP position range $(Z^\ell, Z^u]$ as a function of the trend μ for different values of the cost parameter γ, when $Z = 100$, $\Pi = 0.02$, and $\sigma = 0.02$.

Performance analysis

LPs’ wealth in Uniswap v3 ETH/USDC

	Average	Standard deviation
Number of transactions per LP	11.5	40.2
Position value performance $(\alpha_T/\tilde{x}_0 - 1)$	-1.64%	7.5%
Fee revenue $(\pi_T/\tilde{x}_0 - 1)$	0.155%	0.274%
Hold time	6.1 days	$ 22.4 days
Width	$ 18.7%	$ 43.2%

Table: LP operations statistics in the ETH/USDC pool using operation data of 5,156 different LPs between 5 May 2021 and 18 August 2022. Performance of the position in the pool and fee revenue are not normalised by the hold time.

Performance analysis: the setup

- LP in the ETHUSDC 0.05% pool between 1 January and 18 August 2022.
- Trading frequency: $\Delta t = 1$ minute.
- Execution costs: For quantity Δy of asset Y bought or sold in the pool, a transaction cost $\Delta y \, Z_t^{3/2} \, / \, \kappa$ is incurred.
- Profitability Π: based on past LT activity.
- Position loss: past realised volatility.

$\Longrightarrow$ Performance can be greatly enhanced with signals / predictions.

Performance analysis: the results

	Position value	Fee revenue	Total performance (net of fees)
Optimal strategy	−0.015%	0.0197%	0.0047%
	(0.0951%)	(0.005%)	(0.02%)
Market	−0.0024%	0.0017%	−0.00067%
	(0.02%)	(0.005%)	(0.02%)
Hold	n.a.	n.a.	−0.00016%
			(0.08%)

Table: Mean and standard deviation of the one-minute performance of the LP strategy and its components.

Performance analysis: the results

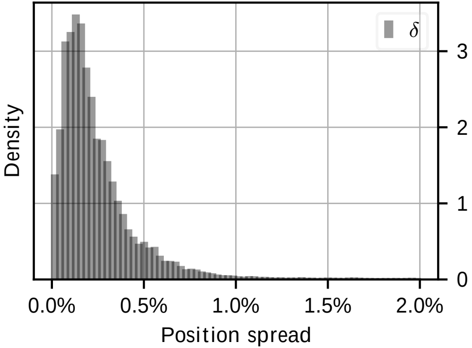


Figure: Distribution of the position spread δ.

Thank you for listening!

Any questions?

Performance analysis: Gas fees & LT activity

- Gas fees: 30.7 USD to provide liquidity, 24.5 USD to withdraw liquidity, and 29.6 USD to take liquidity.

  $\Longrightarrow \tilde{x}_0 > 1.8 \times 10^6$ USD to be profitable.

- However, LT activity limits the performance.

- LP activity profitable in pools with low volatility, increased LT activity, and low gas fees.

Performance analysis: passive versus active strategy

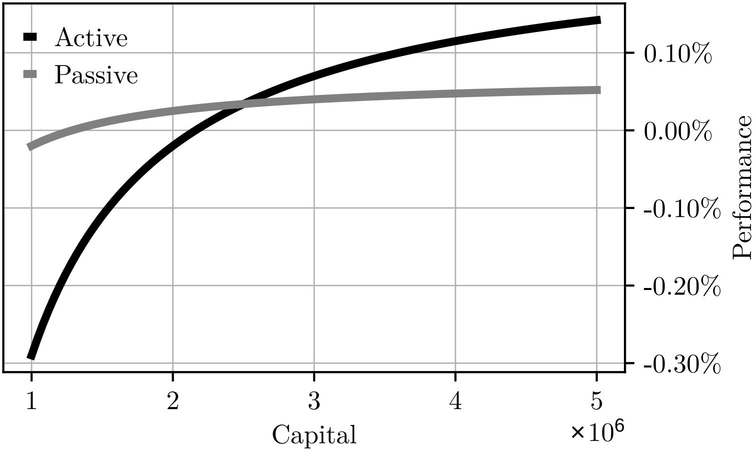


Figure: Profitability of the active strategy and the passive strategy for the ETHUSDC 0.05% pool, as a function of the initial capital.

Assumption 3: asymmetry

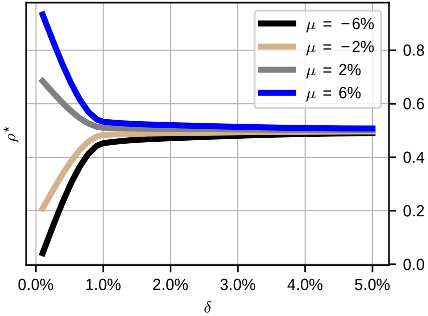


Figure: Optimal position asymmetry $\rho^\star$ as a function of the spread δ of the position, for multiple values of the drift μ.

$$\rho_t = \rho\left(\delta_t, \mu_t\right) = \frac{1}{2} + \frac{\mu_t}{\delta_t} = \frac{1}{2} + \frac{\mu_t}{\delta_t^u + \delta_t^\ell}, \quad \forall t \in [0, T].$$