Equilibrium Liquidity and Risk Offsetting in Decentralised Markets

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ABSTRACT

We develop an economic model of decentralised exchanges (DEXs) in which risk-averse liquidity providers (LPs) manage risk in a centralised exchange (CEX) based on preferences, information, and trading costs. Rational, risk-averse LPs anticipate the frictions associated with replication and manage risk primarily by reducing the reserves supplied to the DEX. Greater aversion reduces the equilibrium viability of liquidity provision, resulting in thinner markets and lower trading volumes. Greater uninformed demand supports deeper liquidity, whereas higher fundamental price volatility erodes it. Finally, while moderate anticipated price changes can improve LP performance, larger changes require more intensive trading in the CEX, generate higher replication costs, and induce LPs to reduce liquidity supply.

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Decentralised exchanges (DEXs) organise trading on blockchains and have become a central component of decentralised finance.[1] Their widespread adoption hinges on whether liquidity provision remains viable when DEXs operate alongside competing centralised exchanges (CEXs), where prices form and liquidity providers (LPs) actively manage risk. Yet, the extant literature abstracts from key economic mechanisms when assessing LP returns and risks: it treats liquidity supply and market conditions as exogenous and focuses on perfect replication in a frictionless CEX. This paper studies the endogenous viability of DEX liquidity provision and the resulting market outcomes when risk-averse LPs anticipate (i) managing exposure in a CEX, accounting for risk preferences, private information, and trading costs, and (ii) strategic interactions with liquidity takers (LTs) whose trading volumes adjust to the level of liquidity supplied.

Our main finding is that a rational, risk-averse LP anticipates the frictions associated with risk offsetting in the CEX and manages risk exposure not only through replication, but primarily by reducing the level of reserves supplied to the DEX. The intuition is as follows. Both (i) net inventory exposure and (ii) trading costs incurred in the CEX generate disutility for a risk-averse LP. Disutility from inventory risk incentivises the LP to actively replicate her DEX position in the CEX, while disutility from CEX trading costs discourages such replication. The equilibrium outcome reflects the balance between these two forces: the ratio of risk aversion to trading costs determines the aggressiveness of replication in the CEX and, in turn, the level and profitability of liquidity provision in the DEX. We find that the viability of liquidity provision in DEXs deteriorates as the disutility from risk aversion dominates that from trading costs, because this leads the LP to trade more heavily on the CEX, and to supply less liquidity due to increasing anticipated trading costs.[2] In some cases, there exists a threshold level of aversion beyond which liquidity provision in DEXs is no longer viable and markets shut down.

Our second finding is that access to private information about future prices does not systematically translate into more profitable liquidity provision. For moderate expected price innovations, the LP benefits from her informational advantage. However, when a risk-averse LP expects large price movements, she anticipates that replicating the position in the CEX will require more intensive trading and higher costs. Anticipating these frictions, the LP supplies less liquidity in the DEX, resulting in thinner markets, lower profitability of liquidity provision, and lower trading volumes of uninformed demand.

[1] Monthly trading volumes on DEXs reached \$420 billion in 2025; see Cong and He (2019); John et al. (2023); Harvey et al. (2024).

[2] A limiting case corresponds to perfect replication, which yields the lowest liquidity supply in our model. In contrast, the extant literature focuses on this case under the assumption of a frictionless CEX; see, for example, Milionis et al. (2022); Cartea et al. (2023); Bichuch and Feinstein (2024).

Our third finding is that the viability of liquidity provision is fundamentally driven by the elasticity and profitability of noise demand, and by the volatility of fundamentals. When noise demand increases or becomes less sensitive to trading costs, the LP anticipates higher fee revenue and is willing to bear greater risk. She does so by reducing the aggressiveness of her CEX trading and by increasing her liquidity supply on the DEX. In contrast, higher fundamental price volatility substantially increases expected adverse selection costs. The LP anticipates this by reducing liquidity supply and offsetting risk aggressively in the CEX to maintain expected outcomes consistent with her risk preferences.

Overall, our results show that the risks and rewards of liquidity provision are not summarised by a single measure from exogenous market conditions. Instead, they emerge endogenously and are determined by (i) the LP's risk preferences, (ii) her private information, and (iii) market conditions, including CEX liquidity depth, fundamental volatility, and the elasticity of uninformed liquidity demand.

Our theoretical contribution is to propose an economic model that endogenises the risk-reward trade-off of liquidity provision in DEXs and the trading volumes of liquidity takers, when the liquidity provider has access to a CEX where inventory risk can be offset at a cost. In our model, there are three types of agents: a representative liquidity provider (LP), noise liquidity takers (noise LTs), and arbitrageurs. These agents interact in three stages. In stage one, the LP chooses the amount of reserves to deposit in the DEX. In stage two, the LP determines a dynamic strategy to (partially) offset exposure in the CEX, taking into account costs, risk preferences, and private price information. In stage three, trading begins: noise LTs with elastic demand arrive (unpredictably) at the DEX and optimise their trading volumes, arbitrageurs align the DEX's marginal price with its fundamental value, and the LP executes her strategy. Our model assumes that the DEX operates as a secondary market and does not influence equilibrium outcomes in the CEX. The model is solved recursively, by dynamic programming.

In **stage three**, noise LTs arrive in the DEX at a known intensity and take the current reserves as given to determine their optimal trading volumes. Specifically, they balance the trading costs implied by the LP's reserves in the DEX against their private utility from holding the asset. Trading costs directly depend on the liquidity reserves deposited by the LP. Specifically, in DEXs, liquidity providers deposit capital into a pool that liquidity takers use to execute trades in exchange for a fee. The DEX functions as an algorithmic market maker whose price and liquidity dynamics are determined by the pricing rules encoded in the DEX's smart contract,[3] the amount of capital in the pool, and the trading fee.

[3] A smart contract is a publicly accessible and immutable program running on the blockchain that defines the rules of interaction with the pool for both liquidity providers and liquidity takers.

In **stage two**, the LP determines her optimal CEX risk offsetting strategy for an arbitrary level of liquidity supply. The strategy explicitly accounts for CEX trading costs, investment horizon, net exposure risk aversion, and private information. We employ variational methods to characterise and solve the optimal strategy in the setting of a DEX with an arbitrary convex bonding curve and when the LP's trading activity generates both permanent and transient market impact. We show that the system of forward–backward stochastic differential equations (FBSDEs) characterizing the LP's strategy reduces to a differential Riccati equation (DRE), whose solution exists, is unique, and can be efficiently computed. In the absence of transient impact, we further derive a closed-form solution. The optimal risk-offsetting strategy comprises two components: (i) a *tracking component*, which balances net exposure aversion and CEX trading costs to partially replicate changes in the DEX's liquidity position, and (ii) a *speculative component*, which adjusts the LP's net exposure to exploit private information.

In **stage one**, the LP anticipates that (i) noise LTs are sensitive to the trading costs implied by the level of DEX reserves, (ii) part of her risk will be offset in the CEX, and (iii) arbitrageurs will align the DEX price to its fundamental value. Thus, the LP sets the optimal level of DEX reserves by trading off anticipated losses to arbitrageurs against (i) anticipated fee revenue from the elastic demand of noise LTs and (ii) the effects of her activity in the CEX. We characterise the LP's optimisation problem and show it admits a solution for DEXs with arbitrary convex bonding curves.

Finally, in the case of constant product markets such as Uniswap, and in the absence of transient price impact in the CEX, we derive analytical formulae for the equilibrium trading volumes, liquidity supply, and the returns and risks of liquidity provision.

Literature review. Numerous works explore the microstructure of DEXs. Angeris et al. (2021); Capponi et al. (2023); Cartea et al. (2024a) show that DEXs generate losses for liquidity suppliers. Jaimungal et al. (2023); Cartea et al. (2025) study liquidity taking in DEXs. Lehar and Parlour (2021) describe competition between DEXs and order books. Hasbrouck et al. (2022) show that higher DEX fees increase liquidity supply and reduce trading costs. Bichuch and Feinstein (2022) formalise the axioms governing DEX design. Klein et al. (2023) examine the role of informed liquidity supply in price discovery. Park (2023) discuss the different types of trading costs in DEXs. Malinova and Park (2024) investigate the potential of DEXs to organise equity trading. Cartea et al. (2024b); He et al. (2024) propose DEX designs aimed at mitigating losses for liquidity suppliers. Recent works also examine the optimal behavior of liquidity providers and the optimal dynamic fee structure of DEXs assuming exogenous levels of reserves; see Bergault et al. (2025); Baggiani et al. (2025). In particular, Campbell et al. (2025) also discusses the costs of replication in the CEX. Finally, Capponi et al. (2025); He et al. (2025) characterise the microstructure of DEXs by incorporating the consensus protocols of blockchains.

Our work is related to the literature on algorithmic trading using stochastic control tools; see Cartea et al. (2015), Guéant (2016), and Donnelly (2022). We incorporate trading signals, first introduced in Cartea and Jaimungal (2016b), where they were interpreted as order-flow indicators.[4] Latent models with trading signals were studied in Casgrain and Jaimungal (2019), while a variational approach to solving trading problems involving multiple agents with heterogeneous beliefs was proposed in Casgrain and Jaimungal (2018, 2020); Wu and Jaimungal (2024). Finally, inventory targeting in optimal trading was analysed in Cartea and Jaimungal (2016a) and Bank et al. (2017).

The remainder of this paper proceeds as follows. Section I describes the economic trade-offs faced by liquidity providers in DEXs and introduces the model. Section II solves for the trading volumes of noise LTs in stage three. Section III analyses the replication problem of the LP in stage two. Section IV derives the optimal liquidity supply in stage one. Section V examines the equilibrium reward–risk trade-off in the case of a constant product market such as Uniswap and presents numerical experiments.

I. General features of the model

DEXs operate with liquidity pooling, where available reserves are aggregated in a common pool, and algorithmic rules, hardcoded in smart contracts running on the blockchain, determine execution prices for liquidity takers (LTs) and revenue for liquidity providers (LPs). This section describes the mechanics of price and liquidity in DEXs, and introduces the general features of our model.

Consider a DEX for a pair of assets $\{X, Y\}$, where X is a reference asset used by agents to value their wealth, and Y is a risky asset. Let a representative LP deposit initial reserves X_0 and Y_0 of assets X and Y, respectively, into the DEX pool at time 0. The LP then remains passive until a terminal investment horizon T, i.e., she neither adds to, nor withdraws from, the reserves in the pool. As trading unfolds over a time window $[0, T]$, where $T > 0$, the available reserves in the pool serve as counterparty to LT trades. Consequently, the reserves in both assets X and Y in the DEX evolve dynamically. Let $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ denote the processes describing the evolution of reserves in assets X and Y, respectively.

DEX price and liquidity. The mechanics of DEXs that determine price and liquidity are defined by *iso-liquidity curves*. Once the LP establishes the pool, and provided she remains

[4] A specific application is investigated in Lehalle and Neuman (2019).

passive, the reserves satisfy, for all $t \in [0, T]$,

$$f(X_t, Y_t) = \kappa^2 = f(X_0, Y_0)\,, \tag{1}$$

where $\kappa > 0$ denotes the *liquidity depth* of the DEX, and $f : (0, \infty)^2 \to (0, \infty)$ is the DEX's *trading function.* The trading function f defines, in (1), all combinations of reserves in assets X and Y that leave the LP *indifferent*, i.e., that do not change the liquidity depth κ. For the analysis that follows, we make the following assumptions.

Assumption 1: (i) *$f \in C^3((0, \infty)^2)$ and has strictly positive partial derivatives.*

(ii) *For each $y > 0$, $f(\cdot, y) : (0, \infty) \to (0, \infty)$ is surjective. Thus, for each $\kappa > 0$, the level set $f(x, y) = \kappa^2$ admits a unique solution $x = \varphi(y, \kappa)$.*

(iii) *$R := \frac{\partial_2 f}{\partial_1 f}$ satisfies $R\,\partial_1 R - \partial_2 R > 0$ everywhere, and is decreasing in κ.*

(iv) *$\partial_1 \varphi$ satisfies the limits $\lim_{y \downarrow 0} \partial_1 \varphi(y, \kappa) = -\infty$ and $\lim_{y \uparrow \infty} \partial_1 \varphi(y, \kappa) = 0$.*

Assumption 1(i) implies that the liquidity depth κ increases in the reserves held in the DEX. We refer to φ in Assumption 1(ii) as the *level function.* By the implicit function theorem, and since f has strictly positive partial derivatives by Assumption 1(i), the mapping φ is C^3 on $(0, \infty)^2$. Using (1), and assuming no additional liquidity is supplied nor withdrawn, we express the reserve in the reference asset X as a function of the reserves in the risky asset Y and the liquidity depth κ as

$$X_t = \varphi(Y_t, \kappa). \tag{2}$$

In DEXs, if an LT wishes to buy a quantity Δy of the risky asset, the indifference condition (1), or equivalently (2), determines the amount Δx of the reference asset that she must pay to the DEX, which satisfies

$$X_t + \Delta x = \varphi(Y_t - \Delta y, \kappa).$$

Thus, the execution price obtained by the LT per unit of the risky asset is given by[5]

$$\frac{\Delta x}{\Delta y} = \frac{\varphi(Y_t - \Delta y, \kappa) - X_t}{\Delta y} = \frac{\varphi(Y_t - \Delta y, \kappa) - \varphi(Y_t, \kappa)}{\Delta y}. \tag{3}$$

[5]The execution price here refers to the amount of the reference asset that the LP pays per unit of the risky asset purchased.

Similarly, if an LT wishes to sell a quantity Δy of the risky asset, the execution price is[6]

$$\frac{\Delta x}{\Delta y} = \frac{\varphi(Y_t,\kappa) - \varphi(Y_t + \Delta y,\kappa)}{\Delta y}. \tag{4}$$

Note that as the traded quantity tends to zero, the execution prices to buy and sell the risky asset in (3)–(4) both converge to the execution price for an infinitesimal quantity $-\partial_1\varphi(Y_t,\kappa)$, which we refer to as the *marginal price*. The marginal price serves as a reference price analogous to the midprice in limit order books. In particular, the difference between the marginal price and the execution prices in (3)–(4) quantifies the trading costs associated with executing a given quantity in the DEX. These trading costs are expressed as

$$\frac{\varphi(Y_t - \Delta y,\kappa) - \varphi(Y_t,\kappa)}{\Delta y} + \partial_1\varphi(Y_t,\kappa) \qquad \text{and} \qquad \frac{\varphi(Y_t,\kappa) - \varphi(Y_t + \Delta y,\kappa)}{\Delta y} + \partial_1\varphi(Y_t,\kappa), \tag{5}$$

and they are positive only when φ is convex in the reserves Y_t, which is ensured by Assumption 1(iii).

Assumption 1-3 also guarantees that the marginal price $-\partial_1\varphi$ is strictly decreasing in the reserves, because

$$\partial_1\varphi(y,\kappa) = -\frac{\partial_2 f(\varphi(y,\kappa),y)}{\partial_1 f(\varphi(y,\kappa),y)} = -R(\varphi(y,\kappa),y),$$

and

$$\partial_{11}\varphi(y,\kappa) = \partial_1 R(\varphi(y,\kappa),y)\, R(\varphi(y,\kappa),y) - \partial_2 R(\varphi(y,\kappa),y).$$

Thus, as LTs sell (resp. buy) the asset Y to the DEX, the reserves in asset Y increase (resp. decrease) and the marginal price decreases (resp. increases).

Moreover, the convexity of the level function ensures that the trading costs (5) are increasing in the quantity Δy bought or sold by the LT. This is akin to limit order books where the cost of walking the book increases with the traded quantity. Finally, Assumption 1(iii) imply that the costs in (5) are decreasing in the liquidity depth κ, so lower levels of reserves make trading more expensive for LTs. This property is central to the trade-offs faced by LPs in DEXs: higher reserve levels reduce trading costs for LTs and attract organic, profitable order flow. However, as discussed below, they also increase the LP's exposure to adverse selection costs.

[6] The execution price here refers to the amount of the reference asset that the LP receives per unit of the risky asset sold.

Liquidity provision rewards. In addition to the liquidity costs arising from the convexity of the level function, LTs also pay a proportional fee $\pi \in (0, 1)$ to LPs when transacting in the DEX. Specifically, for a desired buy quantity Δy of asset Y, an additional amount $\pi \, \Delta y \, F_t$ of the reference asset is paid to LPs. Similarly, for a desired sell quantity Δy , a portion $\pi \, \Delta y \, F_t$ of the amount received from the DEX is kept by LPs. Thus, liquidity-taking activity generates fee revenue for LPs and incentivises increasing the reserves supplied to the DEX.

Liquidity position. Next, we describe the dynamics of the LP's reserves in DEXs. In the remainder of this paper, we work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions. Denote by $(F_t)_{t\geq 0}$ the fundamental price of the risky asset in units of the reference asset X. We assume that the price F follows the stochastic differential equation (SDE)

$$\mathrm{d}F_t = A_t \, F_t \, \mathrm{d}t + \sigma \, F_t \, \mathrm{d}W_t, \tag{6}$$

where $F_0 > 0$ is known, W is an $\mathbb{F}$-Brownian motion, $\sigma > 0$ is a volatility parameter, and $A = (A_t)_{t\in[0,T]}$ is a progressively measurable process satisfying $\mathbb{E}\left[\int_0^T |A_t|^p \, \mathrm{d}t\right] < \infty$ for some $p > 2$. In our model, the process A represents the LP's stochastic private signal.[7]

In this work, we assume arbitrageurs continuously align the pool's marginal price $-\partial_1\varphi(Y_t, \kappa)$ with the fundamental value F_t so

$$F_t = -\partial_1\varphi(Y_t, \kappa).$$

Assumption 1(iv) ensures that $-\partial_1\varphi(\cdot, \kappa)$ is a C^2-diffeomorphism from $(0, \infty)$ to $(0, \infty)$, and therefore admits an inverse $h(\cdot, \kappa)$ which is C^2 on $(0, \infty)$, so[8]

$$F_t = -\partial_1\varphi(Y_t, \kappa) \iff Y_t = h(F_t, \kappa). \tag{7}$$

By Itô's formula, the dynamics of the value of the DEX reserves in units of the reference asset X are

$$\begin{aligned} d\,(X_t + Y_t \, F_t) &= d\,(\varphi(Y_t, \kappa) - Y_t \, \partial_1\varphi(Y_t, \kappa)) \\ &= Y_t \, \mathrm{d}F_t - \underbrace{\tfrac{1}{2}\partial_{11}\varphi(h(F_t, \kappa), \kappa) \; (\partial_1 h(F_t, \kappa))^2 \; \sigma^2 \, F_t^2 \, dt}_{\text{LVR, convexity cost}} \, . \end{aligned} \tag{8}$$

[7] The private signal of the LP may be observable, partially observable, or fully latent. Examples include filtering the LT order flow or using price predictors.

[8] Here, F satisfies the SDE (6), whose solution is $F_t = F_0 \, \exp\left\{\int_0^t (A_s - \frac{\sigma^2}{2}) \, \mathrm{d}s + \sigma \, W_t\right\}$, so the equality (7) is well defined.

The term $Y_t \, dF_t$ on the right-hand side of (8) is commonly regarded as the source of risk for a liquidity position with exogenously fixed initial reserves Y_0. LPs who short a portfolio in a frictionless CEX that fully replicates their position Y_t in the DEX are subject to the negative and predictable loss term on the right-hand side of (8). This term, known as the loss-versus-rebalancing (LVR) or convexity cost, is commonly interpreted as a measure of adverse selection costs in DEXs, which must be compensated by rewards in the form of fee revenue; see Milionis et al. (2022).

In particular, the expected losses to arbitrageurs in (8) are increasing in the depth of liquidity κ and the volatility σ. Thus, the adverse selection component incentivises LPs to reduce the reserves they provide to the DEX.

The model. In contrast to existing characterisations of the viability of liquidity provision, this paper determines the endogenous distribution of LP performance when the LP offsets all or part of her risk in a CEX, at a cost and according to her risk preferences and private information. We also characterise the associated equilibrium depth of liquidity in the DEX and the resulting trading volumes.

The following sections introduce and solve a three-stage model that captures the strategic interactions between LPs and LTs in a DEX. In Stage one, the LP chooses the optimal level of reserves to deposit in the DEX. In Stage two, the LP determines her optimal replication strategy in the CEX. In Stage three, arbitrageurs and noise LTs trade in the DEX.

We solve the model by backward induction. Section II solves stage three, where LTs take the liquidity depth κ as given and determine their optimal trading volumes by balancing DEX trading costs and utility from transacting. These volumes in turn generate fee revenue earned by the LP. Section III solves stage two, where the LP takes the liquidity depth κ as given and determines her optimal CEX replication strategy to balance (i) replication penalties scaled by the LP's risk aversion, (ii) CEX trading costs, and (iii) private signals. Finally, Section IV solves stage one, where the LP anticipates the effects of her trading in the CEX and the activity of both arbitrageurs and noise LTs, to determine the optimal level of DEX reserves.

II. Stage three: trading volumes

A. Assumptions

The timing of stage three corresponds to the LP's investment window $[0, T]$. Throughout this window, two types of LTs interact with the DEX. First, arbitrageurs continuously align the pool's price $-\partial_1 \varphi(Y_t, \kappa)$ with the fundamental value F_t; for simplicity, we do not account for the fee revenue generated by their activity. Second, LTs with elastic demand for the asset trade against the pool. We assume that demand is symmetric, i.e., the number of buyers equals the number of sellers in expectation.

Assume an LT arrives to the DEX at time t, and that her private utility for the asset is V. If $V > 0$ and the LT wishes to buy a quantity $\delta > 0$ of asset Y, her execution costs consist of (i) the execution costs (3) implied by the liquidity supply κ and (ii) the fees $\pi\,\delta\,F_t$ paid to LTs. Thus, the execution price is

$$\frac{\varphi(Y_t - \delta, \kappa) - \varphi(Y_t, \kappa) + \pi\,\delta\,F_t}{\delta}.$$

In our model, noise LTs use the following second-order approximation of the execution price:

$$\begin{aligned}\frac{\varphi(Y_t - \delta, \kappa) - \varphi(Y_t, \kappa) + \pi\,\delta\,F_t}{\delta} &\approx \frac{-\delta\,\partial_1\varphi(Y_t, \kappa) + \frac{1}{2}\delta^2\partial_{11}\varphi(Y_t, \kappa) + \pi\,\delta\,F_t}{\delta} \\ &= F_t + \pi\,F_t + \tfrac{1}{2}\delta\,\partial_{11}\varphi(Y_t, \kappa)\,.\end{aligned} \tag{9}$$

As shown in Cartea et al. (2025); Drissi (2023), this approximation is accurate in practice.[9] In particular, the approximation captures the key economic effect that execution prices worsen as liquidity depth κ decreases, because the convexity term $\partial_{11}\varphi$ is decreasing in κ by Assumption 1(iii).

Similarly, if $V < 0$ and the LT wishes to sell the quantity $\delta > 0$ of asset Y, her execution price is

$$\frac{\varphi(Y_t, \kappa) - \varphi(Y_t + \delta, \kappa) - \pi\,\delta\,F_t}{\delta} \approx F_t - \pi\,F_t - \tfrac{1}{2}\delta\,\partial_{11}\varphi\left(Y_t, \kappa\right)\,. \tag{10}$$

B. Liquidity needs

Noise LTs have random liquidity needs and take the liquidity depth κ in the DEX, determined by the LP in stage 1, as given. To model the random liquidity needs of an LT arriving at time $t \in [0, T]$, we assume that she has a private utility for holding the asset in the form of a private valuation of the risky asset. In our model, the noise LT's utility from holding one unit of the risky asset is $F_t\,(1+V)$, where V is the realization of a random variable symmetrically distributed around zero and independent of all other processes. Specifically, we assume that the distribution of $|V|$ is supported on the compact interval $[\pi, 1]$, and we

[9]Mathematically, the approximation in (9) reduces the LT's problem to a linear-quadratic optimisation problem with an analytical solution.

denote

$$v = \mathbb{E}[|V|] .$$

Thus, an LT observing $V \geq \pi$ (resp. $V \leq -\pi$) wishes to buy (resp. sell) the asset. We assume that the proportional utility $|V|$ exceeds π to ensure positive trading volumes. Moreover, note that $\mathbb{E}[V] = 0$, so the expected cumulative trading volume of noise LTs, from the perspective of the LP, is zero.

C. Trading volumes

If an LT arrives at the DEX at time t, then she determines her optimal trading volume $\delta_t^\star$ by trading off execution costs (9)–(10) against her private utility for the asset. Specifically, the noise LT's performance criterion, when buying or selling a quantity $\delta > 0$, is given by

$$\delta \left(|V| - \pi\right) F_t - \frac{1}{2}\, \delta^2\, \partial_{11}\varphi\left(Y_t, \kappa\right) ,$$

which is maximised with

$$\delta_t^\star = F_t\, \frac{|V| - \pi}{\partial_{11}\varphi\left(Y_t, \kappa\right)} .$$

The trading volume of a noise LT can be written as a function of (i) the liquidity depth κ and (ii) the current level of reserves Y_t, both of which determine the convexity of the level function. Accordingly, we write

$$\delta_t^\star = \delta^\star(Y_t, \kappa) = \frac{|V| - \pi}{\partial_{11}\varphi(Y_t, \kappa)}\, \partial_1\varphi(Y_t, \kappa) .$$

Using the equivalence (7), we may also express noise LT trading volumes as a function of the depth κ and the fundamental price F_t:

$$\delta_t^\star = \delta^\star(F_t, \kappa) = \frac{|V| - \pi}{\partial_{11}\varphi\left(h(F_t, \kappa), \kappa\right)}\, F_t .$$

We assume that, throughout the time window $[0, T]$, the number of noise LTs arriving to the DEX follows a Poisson process $(N_t)_{t\in[0,T]}$ with constant intensity λ. Noise LTs therefore generate fee revenue at a stochastic rate, and the LP's anticipated expected fee revenue in stage one is

$$\mathbb{E}\left[\int_0^T \pi\, \delta_t^\star\, F_t\, dN_t\right] = \lambda\, \pi\, (v - \pi)\, \mathbb{E}\left[\int_0^T \frac{F_t^2}{\partial_{11}\varphi\left(h(F_t, \kappa), \kappa\right)}\, dt\right] .$$

We define the instantaneous rate of fee revenue from the perspective of the LP, and expressed in units of the reference asset X, as a function of the fundamental price and liquidity depth:

$$\Pi_t = \Pi(F_t, \kappa) = \frac{\lambda\, \pi\, (v - \pi)\, F_t^2}{\partial_{11}\varphi\, (h(F_t, \kappa), \kappa)}\,. \tag{11}$$

The key economic force implied by the trading volumes (11) is that greater liquidity depth attracts larger trading volumes because convexity costs are lower, thereby generating higher fee revenue for LPs. In Stage 1, the LP anticipates that supplying more liquidity increases fee income. However, as discussed below, higher liquidity also amplifies losses to arbitrageurs.

III. Stage two: risk offsetting in the centralised exchange

In this section, the LP takes as given the liquidity deposit κ determined in stage one. The liquidity position in the DEX is exposed to adverse selection costs, which increase with market volatility. To manage the risk of her position and to exploit private information, the LP trades in the CEX to maximise her total wealth accross the DEX and the CEX, subject to risk constraints and trading costs.

A. *Assumptions*

In our model, the LP deposits reserves (X_0, Y_0) at time 0 into a DEX characterised by a strictly convex level function φ, and withdraws liquidity at a terminal time $T > 0$. We assume that the LP remains passive over this interval.[10] The risky asset is also traded on a CEX. The LP earns fee revenue from noise LTs trading in the DEX and manages the risk exposure of her DEX position by trading in the CEX at rate $\nu = (\nu_t)_{t\in[0,T]}$. Moreover, the LP also exploits private information driving the fundamental price.

The risky asset's mid-price $S^\nu = (S^\nu_t)_{t\in[0,T]}$ in the CEX has two components: the fundamental price F and a transient market impact $I^\nu = (I^\nu_t)_{t\in[0,T]}$ induced by the LP's trades in the CEX. Formally,

$$S^\nu_t = F_t + I^\nu_t, \qquad t \in [0, T].$$

[10] Active and high-frequency adjustments of liquidity positions are impractical on blockchains: such rebalancing would incur prohibitive gas fees, and on-chain transactions are exposed to predatory *bots* that exploit transaction public visibility.

The transient impact process I^ν satisfies

$$I_t^\nu = \int_0^t \left(c\,\nu_s - \beta\, I_s^\nu\right)\,\mathrm{d}s\,, \tag{12}$$

so that

$$I_t^\nu = c\int_0^t e^{\beta\,(s-t)}\,\nu_s\,\mathrm{d}s\,.$$

Here, $c > 0$ measures the linear price of the LP's trades, and $\beta > 0$ is the resilience parameter governing the decay of transient impact.

By Itô's formula, the LP's DEX reserves in asset Y follow the dynamics

$$\begin{aligned}
\mathrm{d}Y_t &= \partial_1 h(F_t,\kappa)\,\mathrm{d}F_t + \tfrac{1}{2}\,\partial_{11}h(F_t,\kappa)\,\mathrm{d}\langle F\rangle_t \\
&= \left(\partial_1 h(F_t,\kappa)\,A_t\,F_t + \tfrac{\sigma^2}{2}\,\partial_{11}h(F_t,\kappa)\,F_t^2\right)\,\mathrm{d}t + \sigma\,\partial_1 h(F_t,\kappa)\,F_t\,\mathrm{d}W_t \\
&= G_t\,F_t\,\mathrm{d}t + \sigma\,\partial_1 h(F_t,\kappa)\,F_t\,\mathrm{d}W_t\,,
\end{aligned} \tag{13}$$

where we define

$$G_t := \partial_1 h(F_t,\kappa)\,A_t + \tfrac{\sigma^2}{2}\,\partial_{11}h(F_t,\kappa)\,F_t\,. \tag{14}$$

The changes in the reserves in the risky asset Y in (13) are driven by reserves changes due to arbitrageurs continuously aligning the marginal price to its fundamental value.

In our model, we denote the LP's wealth in the pool by $(L_t^\nu)_{t\in[0,T]}$, defined as

$$L_t^\nu := \int_0^t \Pi(F_u,\kappa)\,du + X_t + Y_t\,S_t^\nu.$$

The first term represents the cumulative fee revenue paid by noise LTs, while the second and third terms correspond to the mark-to-market value of the LP's liquidity position valued using the CEX price.

B. *The performance criterion*

The LP holds reserves $\{X_t, Y_t\}$ in the DEX, and her inventory $(Q_t^\nu)_{t\in[0,T]}$ in the CEX is

$$Q_t^\nu = Q_0 + \int_0^t \nu_s\,\mathrm{d}s. \tag{15}$$

Thus, her terminal holdings in the CEX are Q_T^ν, which she values at the terminal CEX price S_T^ν. In our model, the LP maximises her terminal wealth subject to penalties for deviating from a perfect replication strategy, i.e., $Q_t^\nu = -Y_t$. Specifically, the LP's performance criterion, when employing the strategy ν from the admissible set $\mathcal{A}_2$ of $\mathbb{F}$-progressively measurable processes that satisfy $\mathbb{E}\left[\int_0^T |\nu_t|^2 \,\mathrm{d}t\right] < \infty$, is given by

$$\mathbb{E}\left[L_T^\nu + Q_T^\nu\, S_T^\nu - \int_0^T (S_t^\nu + \eta\, \nu_t)\, \nu_t \,\mathrm{d}t - \tfrac{\phi}{2} \int_0^T (Q_t^\nu + Y_t)^2 \,\mathrm{d}t\right].$$

Equivalently, by omitting terms that do not depend on ν, the LP's problem is to maximise

$$\mathbb{E}\Bigg[\underbrace{\left(Y_T + Q_T^\nu\right)\, S_T^\nu}_{\text{combined CEX-DEX position}} \underbrace{- \int_0^T \left(S_t^\nu + \eta\, \nu_t\right)\, \nu_t \,\mathrm{d}t}_{\text{risk offsetting}} \underbrace{- \tfrac{\phi}{2} \int_0^T \left(Q_t^\nu + Y_t\right)^2 \,\mathrm{d}t}_{\text{deviation penalty}}\Bigg]. \tag{16}$$

The first term in the performance criterion (16) represents the sum of the terminal values of the LP's inventory in the CEX and her reserves in the DEX. The second term captures the proceeds from trading in the CEX, and the corresponding trading costs incurred by the LP. We model these costs as a quadratic friction term governed by the cost parameter $\eta > 0$, which reflects the depth of liquidity in the CEX. Note that we assume the DEX operates as a secondary market and does not influence equilibrium outcomes in the CEX.

The third term in (16) is a running penalty for deviating from a perfect replication strategy. Here, $\phi > 0$ is a penalty parameter that scales the deviation cost; higher values of ϕ correspond to greater aversion to holding non-zero net exposure between the LP's positions in the DEX and the CEX. As $\phi \to \infty$, the optimal strategy tends to the perfect replication of the DEX's reserves.

The criterion in (16) can be expressed entirely as a running reward under the following set of assumptions, which we adopt in the remainder of the paper.

Assumption 2: (i) *The private signal satisfies* $\mathbb{E}\left[\exp\left(r \int_0^T |A_s| \,\mathrm{d}s\right)\right] < \infty$ *for all* $r \in \mathbb{R}$.
(ii) *For each* $\kappa > 0$, *there exist real numbers* $C_\kappa, q_\kappa, p_\kappa$ *such that, for all* $x > 0$,

$$|h(x,\kappa)| + |\partial_1 h(x,\kappa)| + |\partial_{11} h(x,\kappa)| \leq C_\kappa \left(x^{q_\kappa} + x^{p_\kappa}\right).$$

Examples satisfying Assumption 2(i) include all continuous Gaussian processes, while constant product markets such as Uniswap is an example of a market that fulfills Assumption 2(ii).

Lemma 1: *The following inequalities hold:*

$$\mathbb{E}\left[\sup_{t \leq T} F_t^q\right] < \infty\,, \quad \mathbb{E}\left[\int_0^T F_t^q \,\mathrm{d}t\right] < \infty\,, \quad \forall q \in \mathbb{R}\,,$$

$$\mathbb{E}\left[\sup_{t\le T} Y_t^q\right] < \infty\,, \quad \mathbb{E}\left[\int_0^t Y_t^q\,\mathrm{d}t\right] < \infty\,, \quad \forall q\in[1,\infty)\,,$$

$$\textit{and}\quad \mathbb{E}\left[\int_0^T |G_t|^q\,\mathrm{d}t\right] < \infty\,, \quad \forall q\in[1,p)\,.$$

Proof See Appendix A.A.

The space $\mathcal{A}_2$ is precisely the real Hilbert space $L^2\left(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t\right)$, where $\mathcal{P}$ is the progressive σ-algebra, with the inner product $\langle\nu,\zeta\rangle \coloneqq \mathbb{E}\Big[\int_0^T \nu_t\,\zeta_t\,\mathrm{d}t\Big]$ and the norm $\|\nu\| \coloneqq \langle\nu,\nu\rangle^{1/2}$. Lemma 1 and Assumption 3 immediately imply the following lemma.

Lemma 2: *$F^q\in\mathcal{A}_2$ for all $q\in\mathbb{R}$. Moreover, for all $\kappa>0$ and $q\ge 1$, $h(F,\kappa)^q$, $(\partial_1 h(F,\kappa))^q$, and $(\partial_{11}h(F,\kappa))^q$ are in $\mathcal{A}_2$.*

Use the inequalities

$$\mathbb{E}\Bigg[\int_0^T \left|\int_0^t \nu_s\,\mathrm{d}s\right|^2 \mathrm{d}t\Bigg] \le \mathbb{E}\Bigg[\int_0^T t\int_0^t |\nu_s|^2\,\mathrm{d}s\,\mathrm{d}t\Bigg] \le T^2\,\mathbb{E}\Bigg[\int_0^T |\nu_t|^2\,\mathrm{d}t\Bigg]$$

and

$$\mathbb{E}\Bigg[\int_0^T \left|\int_0^t e^{\beta\,(s-t)}\nu_s\,\mathrm{d}s\right|^2 \mathrm{d}t\Bigg] \le \mathbb{E}\Bigg[\int_0^T t\int_0^t |\nu_s|^2\,\mathrm{d}s\,\mathrm{d}t\Bigg] \le T^2\,\mathbb{E}\Bigg[\int_0^T |\nu_t|^2\,\mathrm{d}t\Bigg]\,,$$

to define the two bounded linear operators $\mathfrak{Q}\,,\mathfrak{I}:\mathcal{A}_2\to\mathcal{A}_2$ given by

$$(\mathfrak{Q}\nu)_t = \int_0^t \nu_s\,\mathrm{d}s \qquad\text{and}\qquad (\mathfrak{I}\nu)_t = c\int_0^t e^{\beta(s-t)}\,\nu_s\,\mathrm{d}s\,.$$

Notice that $Q^\nu = Q_0+\mathfrak{Q}\nu$ and $I^\nu=\mathfrak{I}\nu$. The following result shows that the performance criterion is a real-valued functional on $\mathcal{A}_2$.

Lemma 3: *Let G be defined in (14). The performance criterion (16) can be written as*

$$J[\nu]+H\,,$$

where J is a linear functional on $\mathcal{A}_2$, and

$$H = (Y_0+Q_0)\,F_0 + \left[\int_0^T \left\{\left(G_t+\sigma^2\,\partial_1 h(F_t,\kappa)\right)\,F_t^2 + (Y_t+Q_0)\,A_t\,F_t - \tfrac{\phi}{2}\,(Y_t+Q_0)^2\right\}\,\mathrm{d}t\right]$$

is a well-defined real number which does not depend on ν. Moreover, J takes the linear-

quadratic form

$$J[\nu] = -\frac{1}{2}\,\mathcal{Q}(\nu) + \mathcal{L}\nu\,, \tag{17}$$

where $\mathcal{Q} : \mathcal{A}_2 \times \mathcal{A}_2 \to \mathbb{R}$ *is the quadratic form defined by*

$$\mathcal{Q}(\nu) = 2\,\eta\,\|\nu\|^2 + 2\,\langle \mathfrak{Q}\nu, \beta\,\mathfrak{I}\nu - c\,\nu\rangle + \phi\,\|\mathfrak{Q}\nu\|^2\,,$$

and $\mathcal{L} : \mathcal{A}_2 \to \mathbb{R}$ *is the bounded linear functional defined by*

$$\mathcal{L}\nu = \langle G\,F, \mathfrak{I}\nu\rangle + \langle Y + Q_0, c\,\nu - \beta\,\mathfrak{I}\nu - \phi\,\mathfrak{Q}\nu\rangle + \langle A\,F, \mathfrak{Q}\nu\rangle\,.$$

Proof See Appendix A.B.

C. *The optimal risk offsetting strategy*

In the remainder of this work, we make the following standing assumption.

Assumption 3: $c < \sqrt{2\,\eta\,\phi}$.

This assumption bounds the instantaneous impact of the LP's trades on CEX prices and ensures that such impacts are offset by sufficiently high trading costs and deviation penalty. This prevents degenerate strategies that would otherwise push prices to infinity. Assumption 3 is not very restrictive, as the parameter ϕ is typically large to reflect the LP's preference for strategies that closely replicate the LP's position in the DEX. Moreover, trading costs η associated with walking the book in the CEX are typically of a larger order of magnitude than the impact parameter c.

We take a variational approach to characterize the optimal replication strategy. To this end, we obtain the following results:

Proposition 1: *Define the symmetric bounded linear operator* $\Lambda : \mathcal{A}_2 \to \mathcal{A}_2$ *by*

$$\Lambda \coloneqq 2\,\eta + \beta\,(\mathfrak{I}^\top\mathfrak{Q} + \mathfrak{Q}^\top\mathfrak{I}) - c\,(\mathfrak{Q} + \mathfrak{Q}^\top) + \phi\,\mathfrak{Q}^\top\mathfrak{Q} \tag{18}$$

and $b \in \mathcal{A}_2$ *by*

$$b \coloneqq \mathfrak{I}^\top(G\,F) + (c - \beta\,\mathfrak{I}^\top - \phi\,\mathfrak{Q}^\top)(Y + Q_0) + \mathfrak{Q}^\top(A\,F)\,. \tag{19}$$

Then the objective J *defined in Lemma 3 satisfies*

$$J[\nu] = -\frac{1}{2}\,\langle \Lambda\nu, \nu\rangle + \langle b, \nu\rangle\,.$$

Proof See Appendix A.C.

Proposition 2: *Λ defined in (18) is* coercive, *i.e., there exists a constant $C > 0$ such that*

$$\langle \Lambda\nu, \nu\rangle \geq C\,\|\nu\|^2\,,$$

for all $\nu \in \mathcal{A}_2$. Consequently, Λ has an inverse, which is also a bounded linear functional on $\mathcal{A}_2$. Moreover, The objective J defined in Lemma 3 is strictly concave.

Proof See Appendix A.D.

Proposition 3: *The objective J defined in Lemma 3 is Gâteaux differentiable, and its Gâteaux derivative $\mathfrak{D}J[\nu]$ at $\nu \in \mathcal{A}_2$ is an element of $\mathcal{A}_2$, given by*

$$\begin{aligned}\mathfrak{D}J[\nu]_t = -2\,\eta\,\nu_t + c\,(Y_t + Q_t^\nu) + \mathbb{E}\left[\int_t^T (A_s\,F_s + c\,\nu_s - \beta\,I_s^\nu - \phi\,(Y_s + Q_s^\nu))\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right] \\ + c\,e^{t\,\beta}\,\mathbb{E}\left[\int_t^T e^{-s\,\beta}\,(G_s\,F_s - \beta\,(Y_s + Q_s^\nu))\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right]\,.\end{aligned} \tag{20}$$

Proof See Appendix A.E.

Theorem 1: *The Gâteaux derivative (20) vanishes at $\nu^\star \in \mathcal{A}_2$ if and only if $\nu^\star$ solves the FBSDE*

$$\begin{cases} 2\,\eta\,\mathrm{d}\nu_t^\star &= (-A_t\,F_t + \beta\,I_t + (\phi + c\,\beta)\,(Y_t + Q_t) + c\,\beta\,Z_t)\,\mathrm{d}t + \mathrm{d}M_t, \quad 2\,\eta\,\nu_T^\star = c\,(Y_T + Q_T)\,, \\ \mathrm{d}Z_t &= (\beta\,(Z_t + Y_t + Q_t) - G_t\,F_t)\,\mathrm{d}t + \mathrm{d}N_t, \quad Z_T = 0\,, \\ \mathrm{d}I_t &= (c\,\nu_t^\star - \beta\,I_t)\,\mathrm{d}t, \quad I_0 = 0\,, \\ \mathrm{d}Q_t &= \nu_t^\star\,\mathrm{d}t\,, \end{cases} \tag{21}$$

for some $\mathbb{F}$-martingales M and N such that $M_T, N_T \in L^2(\Omega)$.

Proof See Appendix A.F.

The next result shows that the solution to the replication problem in the general case reduces to the solution of a differential Riccati equation, whose solution exists, is unique, and can be obtained efficiently numerically.

Proposition 4: *Let*

$$B_{11} = \begin{pmatrix} -\beta & 0 \\ 0 & 0 \end{pmatrix}, \quad B_{12} = \begin{pmatrix} c & 0 \\ 1 & 0 \end{pmatrix}, \quad B_{21} = \frac{1}{2\,\eta} \begin{pmatrix} \beta & \phi + c\beta \\ 0 & 2\,\eta\,\beta \end{pmatrix}, \quad B_{22} = \frac{1}{2\,\eta} \begin{pmatrix} 0 & c\,\beta \\ 0 & 2\,\eta\,\beta \end{pmatrix},$$

$$b_t = \frac{1}{2\,\eta} \begin{pmatrix} -A_t F_t + (\phi + c\beta) Y_t \\ 2\,\eta\,(\beta\,Y_t - G_t\,F_t) \end{pmatrix}, \quad G = \frac{1}{2\,\eta} \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 \\ Q_0 \end{pmatrix}, \quad L = \frac{1}{2\,\eta} \begin{pmatrix} c\,Y_T \\ 0 \end{pmatrix}.$$

Suppose there exists a solution P, which is an $\mathbb{R}^{2\times 2}$-valued C^1 function, to the DRE

$$P'(t) + P(t)\,B_{11} + P(t)\,B_{12}\,P(t) - B_{21} - B_{22}\,P(t) = 0\,, \tag{22}$$

with terminal condition $P(T) = G$. Define the $\mathbb{R}^2$-valued processes ℓ, Ψ, and Φ as follows:

$$\begin{cases} \ell_t & = e^{-\int_0^t (P(u)\,B_{12} - B_{22})\,\mathrm{d}u}\,\mathbb{E}\left[L - \int_t^T e^{\int_0^s (P(u)\,B_{12} - B_{22})\,\mathrm{d}u}\, b_s\,\mathrm{d}s \;\middle|\; \mathcal{F}_t \right], \\ \Phi_t & = e^{\int_0^t (B_{12}\,P(u) + B_{11})\,\mathrm{d}u} \left(K + \int_0^t e^{-\int_0^s (B_{12}\,P(u) + B_{11})\,\mathrm{d}u}\, B_{12}\,\ell_s\,\mathrm{d}s \right), \\ \Psi(t) & = P(t)\,\Phi_t + \ell_t\,. \end{cases}$$

Then (Φ, Ψ) is a solution to the FBSDE (21) with

$$\Psi_t = \begin{pmatrix} \nu_t^\star \\ Z_t \end{pmatrix}, \quad \Phi_t = \begin{pmatrix} I_t \\ Q_t \end{pmatrix}.$$

Moreover, under Assumption 3, the DRE (22) admits a unique solution.

Proof See Appendix A.G.

Proposition 4 shows that in the general case of a DEX with convex level function, the replication strategy of the LP can be obtained efficiently by solving the associated differential Riccati equation (22).

D. *No transient impact*

Here, we consider the case where the LP's trading activity in the CEX is significantly smaller than the overall market activity, so the LP's transient price impact is negligible. Specifically, we assume $c = 0$, in which case $I^\nu = 0$ for any ν. Under this assumption, the LP's optimisation problem is solved explicitly in the following result.

Proposition 5: *Assume $c = 0$. The optimal risk offsetting strategy in the CEX is*

$$\nu_t = P(t) \left(Q_0 \, \tilde{P}(0,t) + \int_0^t \tilde{P}(s,t) \, \ell_s \, \mathrm{d}s \right) + \ell_t \,, \tag{23}$$

where

$$\ell_t = \tfrac{1}{2\eta} \mathbb{E} \left[\int_t^T \tilde{P}(t,s) \, (A_s \, F_s - \phi \, Y_s) \, \mathrm{d}s \,\middle|\, \mathcal{F}_t \right] , \tag{24}$$

and

$$P(t) = \sqrt{\tfrac{\phi}{2\eta}} \tanh\left(\sqrt{\tfrac{\phi}{2\eta}} (t - T) \right) \qquad \text{and} \qquad \tilde{P}(s,t) = \frac{\cosh\left(\sqrt{\frac{\phi}{2\eta}} (t-T) \right)}{\cosh\left(\sqrt{\frac{\phi}{2\eta}} (s-T) \right)} \,. \tag{25}$$

Proof See Appendix A.H.

IV. Stage one: liquidity supply

In the previous section, we derived the optimal stage two replication strategy $\nu_t^\star$ in the CEX for an arbitrary initial position Q_0 and an arbitrary depth of liquidity κ, corresponding to initial DEX reserves $Y_0 = h(F_0, \kappa)$. To determine the optimal liquidity depth $\kappa^\star$ in stage one, the LP anticipates that (i) she will execute her optimal strategy in the CEX at a cost, (ii) trading volumes will respond endogenously to the level of liquidity she supplies, and (iii) adverse selection losses increase with the amount of liquidity deposited in the DEX.

For simplicity, we assume that the LP starts with a CEX position $Q_0 = -Y_0 = -h(F_0, \kappa)$. This assumption facilitates comparisons of performance and risk across different values of the model primitives: CEX trading costs η, risk aversion ϕ, and the profitability parameters $\{\lambda, \upsilon, \pi\}$.

Let $S_t^\star$, $Q_t^\star$, and $L_t^\star$ be the price, inventory, and DEX wealth when the LP executes the optimal strategy $\nu_t^\star$ in the CEX, where

$$L_t^\star := \int_0^t \Pi(F_u, \kappa) \, \mathrm{d}u + X_t + Y_t \, S_t^\star \,,$$

and Π is defined in (11). In the general case, the optimisation problem of stage one is

$$\sup_{\kappa \in [0, \overline{\kappa}]} \mathbb{E}\left[L_T^\star + Q_T^\star \, S_T^\star - \int_0^T \left(S_t^\star + \eta \, \nu_t^\star \right) \, \nu_t^\star \, \mathrm{d}t - \frac{\phi}{2} \int_0^T \left(Q_t^\star + Y_t \right)^2 \, \mathrm{d}t \right] , \tag{K}$$

where $\overline{\kappa}$ denotes the maximum admissible liquidity depth implied by the LP's budget constraint.

The next results show that the LP's objective is well defined and establish mild conditions under which it is continuous and therefore attains its maximum over the compact set $[0, \overline{\kappa}]$.

Proposition 6: *The LP's objective*

$$\mathbb{E}\left[L_T^{\nu^\star} + Q_T^{\nu^\star}\, S_T^{\nu^\star} - \int_0^T \left(S_t^{\nu^\star} + \eta\, \nu_t^\star\right)\, \nu_t^\star\, \mathrm{d}t - \frac{\phi}{2}\int_0^T \left(Q_t^{\nu^\star} + Y_t\right)^2\, \mathrm{d}t\right] \tag{26}$$

is well-defined and can be written as

$$J[\nu^\star] + \mathbb{E}\left[\int_0^T \left\{\left(\frac{\sigma^2}{2} + \lambda\,\pi\,(\pi - v)\right)\, \partial_1 h(F_t,\kappa)\, F_t^2 + A_t\, F_t\,(Y_t - Y_0) - \tfrac{\phi}{2}\,(Y_t - Y_0)^2\right\}\, \mathrm{d}t\right]$$

for all $\kappa > 0$.

Proof See Appendix A.I.

Proposition 7: *Suppose there exist* $\mathfrak{p}, \mathfrak{q} \in \mathbb{R}$ *and a continuous function* $\mathfrak{C} : (0,\infty) \to (0,\infty)$ *such that*

$$|h(x,\kappa) - h(x,\kappa')| + |\partial_1 h(x,\kappa) - \partial_1 h(x,\kappa')| + |\partial_{11} h(x,\kappa) - \partial_{11} h(x,\kappa')| \le (x^{\mathfrak{p}} + x^{\mathfrak{q}})\ |\mathfrak{C}(\kappa) - \mathfrak{C}(\kappa')|$$

for all $x, \kappa, \kappa' > 0$*. Then the LP's objective (26) is continuous in* κ*.*

Proof See Appendix A.J.

V. Constant product markets

To study the implications of risk offsetting in CEXs for liquidity supply and trading in DEXs, we examine the equilibrium outcomes in constant product markets (CPMs) such as Uniswap. In this setting, the level function is

$$\varphi(Y,\kappa) = \frac{\kappa^2}{Y}, \tag{27}$$

and the corresponding fundamental price and reserves satisfy

$$F_t = -\partial_1\varphi(Y_t,\kappa) = \frac{\kappa^2}{Y_t^2} \qquad \text{and} \qquad Y_t = h(F_t,\kappa) = \frac{\kappa}{\sqrt{F_t}}\,. \tag{28}$$

For simplicity, we assume that the LP is a sufficiently small agent whose trades do not generate transient price impact, that is, we set $c = 0$ in (12).

A. Without private information

Assume that the liquidity provider does not use private information and that the fundamental price evolves according to (6) with $A = 0$. The following result characterises the equilibrium liquidity supply, trading volumes, and the LP's strategy in the CEX. The result below is a special case of Proposition 8 and we omit the proof.

Corollary 1: *Assume the level function (27) of a CPM. Then the liquidity supply $\underline{\kappa}$ when the LP does not offset her risk in the CEX is*

$$\underline{\kappa} = \frac{8\,\gamma\left(1-e^{-\sigma^2 T/8}\right) - \sigma^2\left(1-2e^{-\sigma^2 T/8}\right)}{\phi\left(e^{\sigma^2 T}-1-\frac{16}{3}\left(e^{3\sigma^2 T/8}-1\right)+\sigma^2 T\right)} F_0^{3/2}\,, \tag{29}$$

where we refer to

$$\gamma = \frac{\lambda\,\pi\,(v-\pi)}{2}\,, \tag{30}$$

as the profitability parameter. The equilibrium liquidity supply when the LP offsets her risk in the CEX is

$$\kappa^\star = \underline{\kappa}\,\frac{c}{\sigma^2\,\mathfrak{B}+c}\,, \tag{31}$$

where P and $\tilde{P}$ are defined in (25),

$$\mathfrak{B} = \int_0^T \left(1-\tilde{P}(0,t)\right)\left(e^{3\,\sigma^2 t/8}-1\right)\mathrm{d}t - \beta^2\int_0^T\int_s^T g(s)\,\tilde{P}(s,t)\left(e^{\sigma^2 s}e^{3\,\sigma^2(t-s)/8}-e^{3\,\sigma^2 s/8}\right)\mathrm{d}t\,\mathrm{d}s\,,$$
$$c = e^{\sigma^2 T} + \frac{13}{3} - \frac{16}{3}\,e^{3\,\sigma^2 T/8} + \sigma^2\,T\,,$$

and the function g is

$$g(s) = \frac{1}{\cosh\left(\beta\,(s-T)\right)}\int_s^T \cosh\left(\beta\,(u-T)\right)\,e^{3\,\sigma^2(u-s)/8}\,du\,, \qquad \beta = \sqrt{\frac{\phi}{2\,\eta}}.$$

In addition, the equilibrium trading volumes generate fee revenue at the instantaneous rate (11):

$$\Pi\left(F_t,\kappa^\star\right) = \gamma\,\kappa^\star\sqrt{F_t}\,.$$

Finally, the equilibrium risk-offsetting strategy is in (23), where

$$\ell_t = -\beta^2\,\kappa^\star\,F_t^{-1/2}\,g\left(t\right)\,.$$

Next, we show how model primitives influence market outcomes in CPMs. Namely, we study the effect of CEX trading costs η, risk aversion ϕ, fundamental volatility σ, and profitability γ.

Liquidity supply. In CPMs, the equilibrium liquidity $\kappa^\star$ in (31), when the LP dynamically manages her risk in a CEX, takes the no-CEX liquidity $\underline{\kappa}$ in (29) as a reference, and scales it by the coefficient $\frac{c}{\sigma^2 \mathfrak{B} + c}$.

The reference liquidity $\underline{\kappa}$ does not depend on the trading costs η in the CEX and is decreasing in the aversion parameter ϕ because without access to a CEX, reducing risk exposure is only possible by reducing the size of liquidity supply. In contrast, the scaling coefficient depends on both aversion and CEX costs, and it does so only through their ratio $\beta = \phi/\eta$. Specifically, both aversion and trading costs represent forms of disutility to the LP; see (K). The disutility associated with CEX trading costs discourages active replication of the LP's position, whereas the disutility associated with risk aversion encourages active replication. Ultimately, the ratio of these disutilities determines the equilibrium level of liquidity supply and, as we show below, also shapes the LP's behaviour in the CEX. Figure 1 showcases the liquidity supplies (31) and (29), with and without access to a CEX, as a function of model primitives.

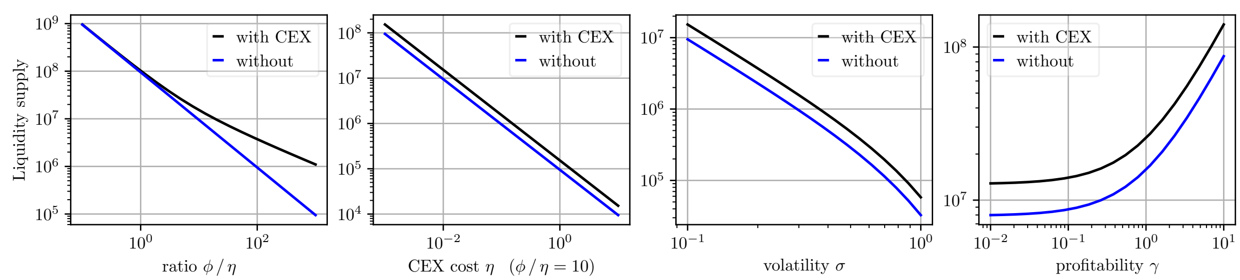
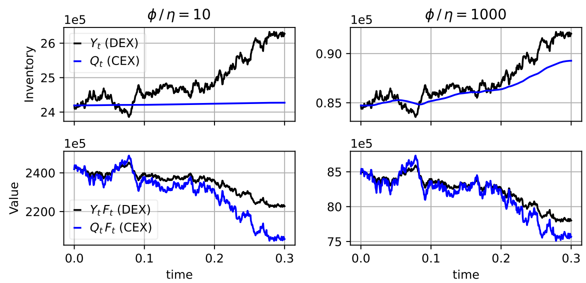


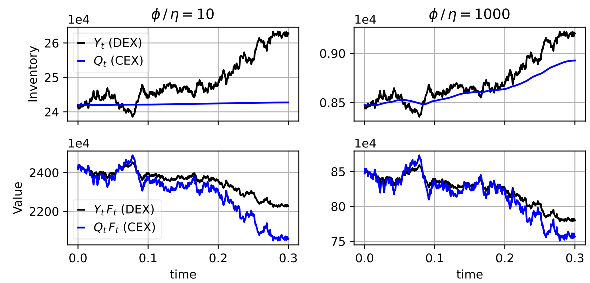
Figure 1. Equilibrium liquidity supply $\kappa^\star$ in (31) (black curves) and reference liquidity $\underline{\kappa}$ in (29) (blue curves), plotted as functions of the model primitives. Default parameter values are: fundamental volatility $\sigma = 0.1$, investment horizon $T = 1$, private signal $A = 0$, CEX trading cost $\eta = 10^{-2}$, ratio $\beta = \phi/\eta = 10$, and profitability $\gamma = 0.2$.

The first panel illustrates the dependence of CPM liquidity according to the ratio β of risk aversion to CEX trading costs (for fixed η). As this ratio increases, the disutility from not closely replicating the DEX position outweighs the disutility generated by CEX trading costs. In this case, the LP more tightly replicates her position in the DEX, as illustrated in more detail in Figure 2a. Moreover, to further decrease the disutility associated with risk exposure, the LP reduces the size of her liquidity supply.

Moreover, the first panel of Figure 1 also shows that the LP increases the scaling applied to the reference liquidity $\underline{\kappa}$ as the ratio β rises. The intuition is as follows. The optimal offsetting strategy effectively reduces the disutility from deviations between CEX and DEX



(a) $\eta = 10^{-2}$



(b) $\eta = 10^{-1}$

Figure 2. Each figure 2a and 2b plots a sample path of the LP's reserves Y_t held in the DEX and the inventory Q_t held in the CEX (top panels), together with their corresponding values expressed in units of the reference asset X (bottom panels). The left panels of each figure correspond to a ratio of risk aversion to trading costs $\beta = 10$, while the right panels correspond to $\beta = 10^3$. Other default parameter values are profitability $\gamma = 0.1$, fundamental volatility $\sigma = 0.2$, and investment horizon $T = 0.3$.

positions, and this benefit becomes increasingly valuable as risk aversion ϕ grows relative to the trading cost η. Anticipating this, the LP applies a higher scaling to the reference liquidity.

The second panel of Figure 1 shows that, for a fixed ratio β, higher trading costs η reduce equilibrium DEX liquidity. The underlying economic force is that dynamic replication in the CEX, at the intensity implied by the ratio β, becomes more costly as η increases. The LP anticipates these higher costs by decreasing her DEX exposure, which reduces the amount of CEX trading required to replicate her position.

This mechanism is further illustrated in Figure 2b. Figures 2a and 2b together show that the degree with which the LP replicates her position in the CEX is governed by the ratio β, while the overall level of liquidity supply decreases as CEX trading costs or aversion increases (holding β fixed).

Finally, the third panel of Figure 1 shows that fundamental price volatility decreases liquidity, and the last panel shows that greater profitability of liquidity demand increases it. In our model, the profitability increases with the fee rate π, the arrival intensity of noise LTs λ, and the average absolute liquidity need v.

Risks and returns. Next, we study the equilibrium risks and returns of liquidity provision in a CPM as a function of model primitives. Specifically, we study the LP's relative change in wealth when she offsets her risk in the CEX, which we compute as follows. Recall that the LP starts with a neutral cumulative CEX–DEX position in asset Y, satisfying $Q_0 + Y_0 = 0$, and with an initial DEX position in the reference asset X equal to $X_0 = \kappa\sqrt{F_0}$. When the

LP executes her optimal CEX strategy, her change in wealth, measured in units of X, is

$$\begin{aligned}
&\int_0^T \Pi(F_t, \kappa^\star)\, dt + X_T + \left(Q_T^\star + Y_T\right) F_T - \int_0^T \left(F_t - \eta\, \nu_t^\star\right) \nu_t^\star\, dt - X_0, \\
&= \underbrace{\int_0^T \Pi(F_t, \kappa^\star)\, dt}_{\text{fee revenue}} + \underbrace{2\, \kappa^\star \left(F_T^{1/2} - F_0^{1/2}\right)}_{\text{DEX position value change}} - \underbrace{\int_0^T Q_t^\star\, dF_t}_{\text{risk offsetting}} - \underbrace{\int_0^T \eta\, \nu_t^{\star\, 2}\, dt}_{\text{CEX cost}}\,,
\end{aligned} \tag{32}$$

where $\nu_t^\star$ is the optimal trading rate in the CEX and $Q_t^\star$ the corresponding inventory. To obtain the relative change in the LP's wealth, we normalise (32) by the initial cash position $X_0 = \kappa^\star \sqrt{F_0}$.

Note that the expected change in the value of the LP's DEX liquidity position is

$$\mathbb{E}\left[2\, \kappa^\star\, (F_T^{1/2} - F_0^{1/2})\right] = F_0^{1/2} \left(e^{-\sigma^2\, T/8} - 1\right)\,, \tag{33}$$

which is always negative. The viability of DEX liquidity provision depends on whether the stage-three fee revenue, adjusted by replication costs and the proceeds form risk offsetting, covers the adverse selection costs (33).

When the LP does not offset her exposure in the CEX, i.e., when $\nu \equiv 0$, her inventory in the CEX remains constant, $Q_t = Q_0$, and the expected change in her wealth is

$$\underbrace{\int_0^T \Pi(F_t, \underline{\kappa})\, dt}_{\text{fee revenue}} + \underbrace{2\, \underline{\kappa} \left(F_T^{1/2} - F_0^{1/2}\right)}_{\text{DEX position value change}} - \underbrace{Q_0\, (F_T - F_0)}_{\text{CEX position}}\,. \tag{34}$$

Comparing (32) and (34) isolates the effect of dynamic risk offsetting in the CEX: it reduces inventory risk at the expense of trading frictions $\eta\, \nu_t^2$, but it may also alter expected fee revenue and terminal payoffs through the adjusted liquidity choice $\kappa^\star$ studied above. Figure 3 illustrates these effects by plotting the distribution of the profit and loss of DEX liquidity provision as a function of model primitives.

Figure 3 highlights a first-order economic effect of risk offsetting on the viability of liquidity provision in DEXs. While the expected adverse selection losses to arbitrageurs in (33) are unaffected by the LP's trading in the CEX, the variance of these losses decreases as the ratio β increases and replication becomes more aggressive. At the same time, the trading costs generated by the LP's activity in the CEX increase with the intensity of replication. Consequently, the viability of DEX liquidity provision is shaped by (i) the LP's aversion to risk, which determine the trading costs incurred in the CEX, and by (ii) fee revenue. In particular, Figure 3 shows that beyond a threshold level of risk aversion, liquidity provision is no longer viable and DEX markets shut down.

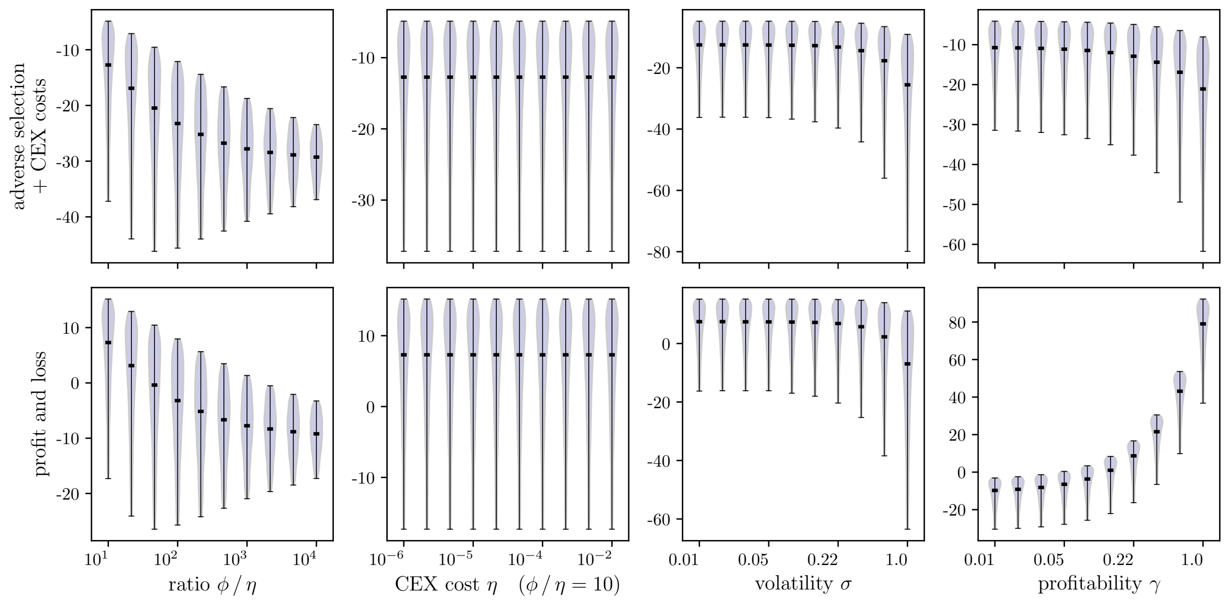


Figure 3. Distribution of the equilibrium adverse selection and trading costs (top panels) and the equilibrium payoff of liquidity provision (bottom panels). The distribution is obtained from 2000 market simulations, with the time interval discretised into 1000 steps. Default parameter values are $\sigma = 0.1$, $T = 1$, $A = 0$, $\eta = 10^{-2}$, $\beta = 10$, and $\gamma = 0.25$.

The second column of panels in Figure 3 shows that the equilibrium percentage returns and risk of liquidity provision depend only on the ratio of risk aversion to trading costs, and not on the absolute level of either parameter. The intuition is that the LP adjusts the aggressiveness of risk offsetting according to the ratio β, while she adjusts the level of liquidity supply according to the absolute level of risk aversion. As a result, returns and risks of liquidity provision, when measured relative to the LP's initial wealth, are driven solely by the ratio β.

The third column of Figure 3 shows that higher fundamental price volatility substantially increases adverse selection costs, thereby undermining the viability of liquidity provision in CPMs. In contrast, the final column illustrates how the profitability of noise demand affects the returns and risks of liquidity provision. As profitability γ increases, the LP is willing to supply more liquidity (see Figure 1) and to bear greater inventory risk, and the incentive to offset large positions at quadratic cost in the CEX diminishes. In equilibrium, although adverse selection losses and inventory risk rise, they are more than compensated by higher fee revenue.

B. *Risk offsetting and private information*

Here, we assume that the LP employs a private signal driving the drift of the fundamental price of asset Y. The equilibrium liquidity supplies, with and without risk offsetting, are characterised in the following result.

Proposition 8: *Assume $Y_t = F_t^{-1/2}\,\kappa$ as in (28). The equilibrium liquidity supply in the CPM when the LP does not use the CEX is*

$$\underline{\kappa} = \frac{\mathbb{E}\left[\int_0^T \gamma\, F_t^{1/2}\,\mathrm{d}t + 2\,F_T^{1/2} - F_0^{-1/2}\,F_T\right]}{\phi\,\mathbb{E}\left[\int_0^T \left(F_t^{-1/2} - F_0^{-1/2}\right)^2\,\mathrm{d}t\right]}\,, \tag{35}$$

Moreover, define the following processes

$$C_t^\ell = -\frac{\phi}{2\,\eta}\,\mathbb{E}\left[\int_t^T \tilde{P}(t,s)\,F_s^{-1/2}\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right], \qquad D_t^\ell = \frac{1}{2\,\eta}\,\mathbb{E}\left[\int_t^T \tilde{P}(t,s)\,A_s\,F_s\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right],$$

$$C_t^Q = -F_0^{-1/2}\,\tilde{P}(0,t) + \int_0^t \tilde{P}(s,t)\,C_s^\ell\,\mathrm{d}s, \qquad \tilde{M}_t = \mathbb{E}\left[\int_0^T \tilde{P}(0,s)\,F_s^{-1/2}\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right],$$

$$D_t^Q = \int_0^t \tilde{P}(s,t)\,D_s^\ell\,\mathrm{d}s, \qquad C_t^\nu = P(t)\,C_t^Q + C_t^\ell, \qquad \text{and} \qquad D_t^\nu = P(t)\,D_t^Q + D_t^\ell,$$

where P and $\tilde{P}$ are defined in (25), and assume that the processes

$$\int_0^t \tilde{P}(s,0)\,D_s^Q\,\mathrm{d}\tilde{M}_s \qquad \text{and} \qquad \int_0^t \tilde{P}(s,0)\,C_s^Q\,\mathrm{d}\tilde{M}_s\,, \quad 0 \le t \le T\,, \tag{36}$$

are martingales. Then the equilibrium supply of liquidity is

$$\kappa^\star = \left(\underline{\kappa} + \frac{\mathfrak{A}}{\phi\,\mathbb{E}\left[\int_0^T \left(F_t^{-1/2} - F_0^{-1/2}\right)^2\,\mathrm{d}t\right]}\right)\frac{\mathbb{E}\left[\int_0^T \left(F_t^{-1/2} - F_0^{-1/2}\right)^2\,\mathrm{d}t\right]}{\mathfrak{B} + \mathbb{E}\left[\int_0^T \left(F_t^{-1/2} - F_0^{-1/2}\right)^2\,\mathrm{d}t\right]}\,, \tag{37}$$

where

$$\mathfrak{A} = \mathbb{E}\left[\int_0^T \left(C_t^Q + F_0^{-1/2}\right)\,A_t\,F_t\,\mathrm{d}t\right], \quad \mathfrak{B} = \mathbb{E}\left[\int_0^T \left(C_t^Q + F_0^{-1/2}\right)\,\left(F_t^{-1/2} - F_0^{-1/2}\right)\,\mathrm{d}t\right].$$

In particular,

$$\mathfrak{B} + \mathbb{E}\left[\int_0^T \left(F_t^{-1/2} - F_0^{-1/2}\right)^2\,\mathrm{d}t\right] \ge 0.$$

Proof Appendix A.K.

The next result shows that Proposition 8 applies to the popular case in which the private signal A follows an Ornstein–Uhlenbeck process with dynamics

$$\mathrm{d}A_t = \theta\,(\mu - A_t)\,dt + \xi\,dW_t\,. \tag{38}$$

Lemma 4: *The processes defined in* (36) *are martingales if A is an Ornstein-Uhlenbeck process with dynamics* (38).

Proof Appendix A.L.

The equilibrium liquidity supply (37) takes the liquidity level (35) as a reference, adjusts it upward or downward depending on the value of the private signal, and then applies a scaling that depends on the ratio β of risk aversion to trading costs. The dependence of liquidity supply on β, trading costs η, volatility σ, and profitability γ is qualitatively similar to that studied in the previous section. Figure 4 considers the simple case of constant signal A and examines how equilibrium liquidity supply varies with the LP's private information.

Figure 4 shows that, counterintuitively, private information does not systematically lead to higher performance or deeper markets. For moderate positive values of the fundamental price drift, the LP anticipates that, in addition to fee revenue, the positive drift will improve performance, and therefore increases liquidity supply relative to the zero-drift benchmark. However, for large absolute values of the signal, the LP anticipates that replicating the position in the CEX will require more intensive trading and generate higher trading costs. Anticipating these costs, she reduces her liquidity supply.

The extent of this reduction increases with the ratio β, as illustrated in Figure 4. When β is large, equilibrium liquidity supply is lower and the LP anticipates reduced CEX trading activity. As a result, the range of signal values A for which liquidity supply exceeds the reference level widens.

VI. Conclusions

This paper builds a structural model of liquidity provision in DEXs in which arbitrageurs align DEX prices with fundamentals, thereby generating adverse selection losses for LPs, while noise and elastic demand generates fee revenue. We show that, once trading volumes and liquidity supply are endogenised, the losses and risks borne by liquidity providers are not summarised by any single measure. Instead, they depend on (i) market conditions, including CEX liquidity depth, fundamental volatility, and noise trading activity, and on (ii) the LP's risk aversion, which ultimately shapes the distribution of returns from DEX liquidity provision.

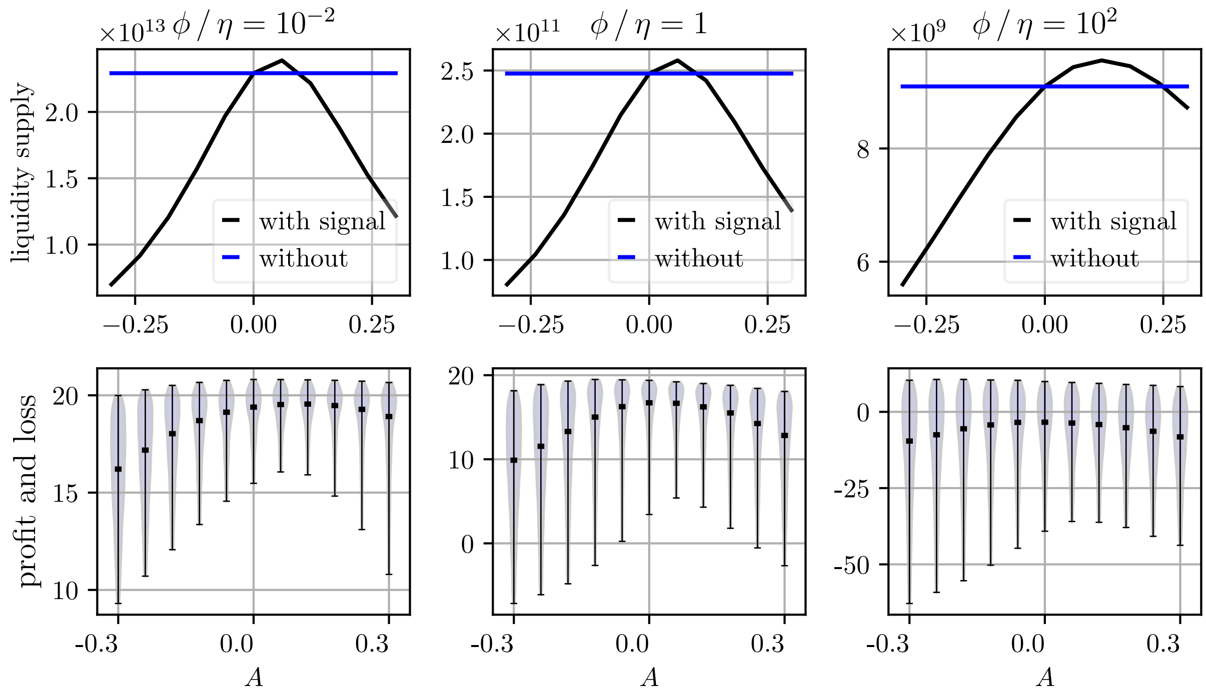


Figure 4. Top panels plot the equilibrium liquidity supply $\kappa^\star$ in (37) as a function of the constant signal A (black curves), together with the equilibrium liquidity supply $\kappa^\star$ evaluated at $A = 0$ (blue curves). The bottom panels show the equilibrium distribution of payoffs from DEX liquidity provision. The left panels correspond to a ratio of aversion to CEX trading costs $\beta = 10^{-2}$, the middle panels to $\beta = 1$, and the right panels to $\beta = 100$. Default parameter values are: fundamental volatility $\sigma = 0.2$, investment horizon $T = 1$, CEX trading cost $\eta = 10^{-6}$, and profitability $\gamma = 0.2$. The distributions in the bottom panels are obtained from 2000 market simulations, with the time interval discretised into 1000 steps.

Appendix A. Proofs

Appendix A. Proof of Lemma 1

For each $q \in \mathbb{R}$, consider the exponential martingale $M(q) = (M(q)_t)_{t\geq 0}$:

$$M(q)_t := e^{q\,\sigma\,W_t - \frac{1}{2}\,q^2\,\sigma^2\,t},$$

and write

$$F_t^q = F_0^q\,e^{\frac{1}{2}\,(q^2-q)\,\sigma^2\,t}\;e^{q\int_0^t A_s\,\mathrm{d}s}\,M(q)_t\,.$$

By Cauchy-Schwarz inequality, Doob's inequality, and Assumption 2-1, we obtain

$$\mathbb{E}\left[\sup_{t\leq T} F_t^q\right] \leq F_0^q\,e^{\frac{1}{2}\,|q^2-q|\,\sigma^2\,T}\,\mathbb{E}\left[e^{|q|\int_0^T |A_s|\,\mathrm{d}s}\;\sup_{t\leq T} M(q)_t\right]$$

$$
\begin{aligned}
&\le F_0^q\, e^{\frac{1}{2}\,|q^2-q|\,\sigma^2\,T}\,\mathbb{E}\left[e^{2\,|q|\int_0^T |A_s|\,\mathrm{d}s}\right]^{1/2}\,\mathbb{E}\left[\sup_{t\le T}(M(q)_t)^2\right]^{1/2}\\
&\le 2\,F_0^q\, e^{\frac{1}{2}\,|q^2-q|\,\sigma^2\,T}\,\mathbb{E}\left[e^{2\,|q|\int_0^T |A_s|\,\mathrm{d}s}\right]^{1/2}\,\mathbb{E}\left[(M(q)_T)^2\right]^{1/2}\\
&\le 2\,F_0^q\, e^{\frac{1}{2}\,(|q^2-q|+q^2)\,\sigma^2\,T}\,\mathbb{E}\left[e^{2\,|q|\int_0^T |A_s|\,\mathrm{d}s}\right]^{1/2}\\
&< \infty
\end{aligned}
$$

and

$$
\mathbb{E}\left[\int_0^T F_t^q\,\mathrm{d}t\right] \le T\,\mathbb{E}\left[\sup_{t\le T} F_t^q\right] < \infty\,.
$$

By Assumption 2-2, we obtain, for all $q \in [1,\infty)$,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\le T} Y_t^q\right] = \mathbb{E}\left[\sup_{t\le T} h(F_t,\kappa)^q\right] &\le C_\kappa^q\,\mathbb{E}\left[\sup_{t\le T}\,(F_t^{q_\kappa}+F_t^{p_\kappa})^q\right]\\
&\le C_\kappa^q\,2^{q-1}\left(\mathbb{E}\left[\sup_{t\le T} F_t^{q_\kappa\,q}\right]+\mathbb{E}\left[\sup_{t\le T} F_t^{p_\kappa\,q}\right]\right)\\
&< \infty\,.
\end{aligned}
$$

Moreover, we also obtain

$$
\mathbb{E}\left[\int_0^t Y_t^q\,\mathrm{d}t\right] \le T\,\mathbb{E}\left[\sup_{t\le T} Y_t^q\right] < \infty\,.
$$

and

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T |G_t|^q\,\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^T \left|\partial_1 h(F_t,\kappa)\,A_t + \tfrac{\sigma^2}{2}\,\partial_{11} h(F_t,\kappa)\,F_t\right|^q\,\mathrm{d}t\right]\\
&\lesssim \mathbb{E}\left[\int_0^T (F_t^{q_\kappa\,q}+F_t^{p_\kappa\,q})\,|A_t|^q\,\mathrm{d}t + \int_0^T \left(F_t^{q_\kappa\,q+q}+F_t^{p_\kappa\,q+q}\right)\mathrm{d}t\right]\\
&\lesssim \mathbb{E}\left[\int_0^T \left(F_t^{\frac{p\,q_\kappa\,q}{p-q}}+F_t^{\frac{p\,p_\kappa\,q}{p-q}}\right)\mathrm{d}t\right]^{\frac{p-q}{p}}\,\mathbb{E}\left[\int_0^T |A_t|^p\,\mathrm{d}t\right]^{\frac{q}{p}} + \mathbb{E}\left[\int_0^T \left(F_t^{q_\kappa\,q+q}+F_t^{p_\kappa\,q+q}\right)\mathrm{d}t\right]\\
&< \infty\,,\quad \forall q\in[1,p)\,.
\end{aligned}
$$

□

Appendix B. Proof of Lemma 3

The proof proceeds in four steps. First, we show that the performance criterion (16) is well-defined and continuous. Next, we show that the functional J in (17) is also well-defined and continuous. Next, we show that the performance criterion (16) and J in (17) agree up to a constant on bounded processes. Finally, we conclude.

Step 1. First, we show that the performance criterion (16) is well-defined and continuous. Take $\nu \in \mathcal{A}_2$.

Then

$$\mathbb{E}\Big[|Q_T^\nu|^2\Big] = \mathbb{E}\left[\left|Q_0 + \int_0^T \nu_t \,\mathrm{d}t\right|^2\right] \le 2\,\left(|Q_0|^2 + T\,\mathbb{E}\left[\int_0^T |\nu_t|^2 \,\mathrm{d}t\right]\right) < \infty$$

and

$$\mathbb{E}\Big[|I_T^\nu|^2\Big] = \mathbb{E}\left[\left|c\int_0^T e^{\beta\,(t-T)}\,\nu_t \,\mathrm{d}t\right|^2\right] \le c^2\,T\,\mathbb{E}\left[\int_0^T |\nu_t|^2 \,\mathrm{d}t\right] < \infty\,.$$

These estimates together with Lemma 1 and Cauchy-Schwarz inequality imply

$$\mathbb{E}[(Y_T + Q_T^\nu)\;S_T^\nu]$$

is well-defined. Because (16) can be written as

$$\mathbb{E}[(Y_T + Q_T^\nu)\;S_T^\nu] - \eta\,\|\nu\|^2 - \langle \mathfrak{I}\nu, \nu\rangle - \frac{\phi}{2}\,\|\mathfrak{Q}\nu\|^2 - \langle F, \nu\rangle - \phi\,\langle Y, \mathfrak{Q}\nu\rangle - \frac{\phi}{2}\,\|Y\|^2\,,$$

where $Y \in \mathcal{A}_2$ by Lemma 1 and $\mathfrak{Q}$ and $\mathfrak{I}$ are bounded linear operators on $\mathcal{A}_2$, it is well-defined.

Write

$$\nu \mapsto -\eta\,\|\nu\|^2 - \langle \mathfrak{I}\nu, \nu\rangle - \frac{\phi}{2}\,\|\mathfrak{Q}\nu\|^2 - \langle F, \nu\rangle - \phi\,\langle Y, \mathfrak{Q}\nu\rangle$$

is a linear-quadratic form on $\mathcal{A}_2$, it is continuous, it remains to show $\mathbb{E}[(Y_T + Q_T^\nu)\;S_T^\nu]$ is continuos in ν. To that end, take $\nu^{(n)} \to \nu$ in $\mathcal{A}_2$. Then

$$\begin{aligned}
\left|\mathbb{E}\left[Y_T\,\left(I_T^{\nu^{(n)}} - I_T^\nu\right)\right]\right| &\le \mathbb{E}\left[|Y_T|^2\right]^{1/2}\,\mathbb{E}\left[\left|I_T^{\nu^{(n)}} - I_T^\nu\right|^2\right]^{1/2} \\
&\le c\,\sqrt{T}\,\mathbb{E}\left[|Y_T|\right]^{1/2}\,\mathbb{E}\left[\int_0^T \left|\nu_t^{(n)} - \nu_t\right|^2\,\mathrm{d}t\right]^{1/2}\,, \\
\left|\mathbb{E}\left[F_T\,\left(Q_T^{\nu^{(n)}} - Q_T^\nu\right)\right]\right| &\le \mathbb{E}\left[|F_T|^2\right]^{1/2}\,\mathbb{E}\left[\left|Q_T^{\nu^{(n)}} - Q_T^\nu\right|^2\right]^{1/2} \\
&\le \sqrt{2\,T}\,\mathbb{E}\left[|F_T|\right]^{1/2}\,\mathbb{E}\left[\int_0^T \left|\nu_t^{(n)} - \nu_t\right|^2\,\mathrm{d}t\right]^{1/2}\,,
\end{aligned}$$

and, by Minkwoski's inequality

$$\begin{aligned}
\left|\mathbb{E}\left[Q_T^{\nu^{(n)}}\,I_T^{\nu^{(n)}} - Q_T^\nu\,I_T^\nu\right]\right| &= \left|\mathbb{E}\left[Q_T^{\nu^{(n)}}\,\left(I_T^{\nu^{(n)}} - I_T^\nu\right) + \left(Q_T^{\nu^{(n)}} - Q_T^\nu\right)\,I_T^\nu\right]\right| \\
&\le \mathbb{E}\left[\left|Q_T^{\nu^{(n)}}\right|^2\right]^{1/2}\,\mathbb{E}\left[\left|I_T^{\nu^{(n)}} - I_T^\nu\right|^2\right]^{1/2} + \mathbb{E}\left[\left|Q_T^{\nu^{(n)}} - Q_T^\nu\right|^2\right]^{1/2}\,\mathbb{E}\left[|I_T^\nu|^2\right]^{1/2} \\
&\le \left(\mathbb{E}\left[\left|Q_T^{\nu^{(n)}} - Q_T^\nu\right|^2\right]^{1/2} + \mathbb{E}\left[|Q_T^\nu|^2\right]^{1/2}\right)\,\mathbb{E}\left[\left|I_T^{\nu^{(n)}} - I_T^\nu\right|^2\right]^{1/2} \\
&\quad + \mathbb{E}\left[\left|Q_T^{\nu^{(n)}} - Q_T^\nu\right|^2\right]^{1/2}\,\mathbb{E}\left[|I_T^\nu|^2\right]^{1/2}
\end{aligned}$$

$$\leq \sqrt{2}\, c\, T\, \mathbb{E}\left[\int_0^T \left|\nu_t^{(n)} - \nu_t\right|^2 \,\mathrm{d}t\right] + c\,\sqrt{T}\,\mathbb{E}\left[|Q_T^\nu|^2\right]^{1/2} \mathbb{E}\left[\int_0^T \left|\nu_t^{(n)} - \nu_t\right|^2 \,\mathrm{d}t\right]^{1/2}$$
$$+ \sqrt{2\,T}\,\mathbb{E}\left[\int_0^T \left|\nu_t^{(n)} - \nu_t\right|^2 \,\mathrm{d}t\right]^{1/2} \mathbb{E}\left[|I_T^\nu|^2\right]^{1/2}.$$

These estimates imply $\mathbb{E}[(Y_T + Q_T^\nu)\, S_T^\nu]$ is continuous in ν, as desired.

Step 2. Next, we show that J is well-defined and continuous. Because $\mathfrak{Q}$ and $\mathfrak{I}$ are bounded linear operators on $\mathcal{A}_2$, the quadratic form $\mathcal{Q}$ is well-defined and continuous. Because we know $F \in \mathcal{A}_2$ by Lemma 1, it remains to show the processes $G\,F$ and $A\,F$ are in $\mathcal{A}_2$. Indeed, if $q \in (2, p)$, then

$$\mathbb{E}\left[\int_0^T |G_t\,F_t|^2 \,\mathrm{d}t\right] \leq \mathbb{E}\left[\int_0^T |G_t|^q \,\mathrm{d}t\right]^{\frac{2}{q}} \mathbb{E}\left[\int_0^T F_t^{\frac{2\,q}{q-2}} \,\mathrm{d}t\right]^{\frac{q-2}{q}} < \infty$$

and

$$\mathbb{E}\left[\int_0^T |A_t\,F_t|^2 \,\mathrm{d}t\right] \leq \mathbb{E}\left[\int_0^T |A_t|^p \,\mathrm{d}t\right]^{\frac{2}{p}} \mathbb{E}\left[\int_0^T F_t^{\frac{2\,p}{p-2}} \,\mathrm{d}t\right]^{\frac{p-2}{p}} < \infty\,,$$

by Lemma 1.

Step 3. Next, we show that the performance criterion (16) and J in (17) agree up to a constant on bounded processes. Take $\nu \in \mathcal{A}_2$ such that $|\nu| \leq N$ for some constant N. Then

$$|Q_t^\nu| = \left|Q_0 + \int_0^t \nu_s \,\mathrm{d}s\right| \leq |Q_0| + T\,N$$

and

$$|I_t^\nu| = \left|c\int_0^t e^{\beta(s-t)}\,\nu_s \,\mathrm{d}s\right| \leq c\,T\,N,.$$

By Itô's formula, (12), (6), (13), and (15), we have

$$\begin{aligned}
(Y_T + Q_T^\nu)\,S_T^\nu &= (Y_0 + Q_0)\,F_0 + \int_0^T (Y_t + Q_t^\nu)\,\mathrm{d}S_t^\nu + \int_0^T S_t^\nu\,\mathrm{d}Y_t + \int_0^T S_t^\nu\,\mathrm{d}Q_t^\nu + \int_0^T \mathrm{d}\langle Y, F\rangle_t \\
&= (Y_0 + Q_0)\,F_0 + \int_0^T (Y_t + Q_t^\nu)\,(A_t\,F_t + c\,\nu_t - \beta\,I_t^\nu)\mathrm{d}t \\
&\quad + \int_0^T (F_t + I_t^\nu)\,G_t\,F_t\,\mathrm{d}t + \int_0^T S_t^\nu\,\nu_t\,\mathrm{d}t + \int_0^T \sigma^2\,\partial_1 h(F_t,\kappa)\,F_t^2\,\mathrm{d}t \\
&\quad + \sigma\int_0^T F_t\,\left[Y_t + Q_t^\nu + \partial_1 h(F_t,\kappa)\,(F_t + I_t^\nu)\right]\mathrm{d}W_t\,,
\end{aligned}$$

so

$$(Y_T + Q_T^\nu)\;S_T^\nu - \int_0^T (S_t^\nu + \eta\,\nu_t)\;\nu_t\,\mathrm{d}t - \tfrac{\phi}{2}\int_0^T (Q_t^\nu + Y_t)^2\;\mathrm{d}t$$

$$= (Y_0 + Q_0)\, F_0 + \int_0^T \Big\{ \big(G_t + \sigma^2\, \partial_1 h(F_t, \kappa)\big)\; F_t^2 + (Y_t + Q_0)\, A_t\, F_t - \tfrac{\phi}{2}\, (Y_t + Q_0)^2 \Big\}\; \mathrm{d}t$$

$$+ \int_0^T \Big\{ I_t^\nu\, G_t\, F_t - \eta\, \nu_t^2 + (Y_t + Q_t^\nu)\; (c\, \nu_t - \beta\, I_t^\nu) + (A_t\, F_t - \phi\, (Y_t + Q_0))\; (Q_t^\nu - Q_0) - \tfrac{\phi}{2}\; (Q_t^\nu - Q_0)^2 \Big\}\; \mathrm{d}t$$

$$+ \sigma \int_0^T F_t\; [Y_t + Q_t^\nu + \partial_1 h(F_t, \kappa)\, (F_t + I_t^\nu)]\; \mathrm{d}W_t\,,$$

where

$$\mathbb{E}\left[\int_0^T \Big| \big(G_t + \sigma^2\, \partial_1 h(F_t, \kappa)\big)\; F_t^2 + (Y_t + Q_0)\, A_t\, F_t - \tfrac{\phi}{2}\, (Y_t + Q_0)^2 \Big|\; \mathrm{d}t \right] < \infty\,.$$

Since

$$\begin{aligned} &\mathbb{E}\left[\int_0^T F_t^2\; |Y_t + Q_t^\nu + \partial_1 h(F_t, \kappa)\, (F_t + I_t^\nu)|^2\; \mathrm{d}t \right] \\ &\lesssim \mathbb{E}\left[\int_0^T F_t^2\; \big(Y_t^2 + |Q_t^\nu|^2 + |\partial_1 h(F_t, \kappa)|^2\; \big(F_t^2 + |I_t^\nu|^2\big)\big)\; \mathrm{d}t \right] \\ &\lesssim \mathbb{E}\left[\int_0^T F_t^2\; \Big(Y_t^2 + (|Q_0| + T\, N)^2 + \Big(F_t^{2\, q_\kappa} + F_t^{2\, p_\kappa}\Big)\; \big(F_t^2 + c^2\, T^2\, N^2\big)\Big)\; \mathrm{d}t \right] \\ &< \infty\,, \end{aligned}$$

the process

$$\int_0^t F_t\; [Y_t + Q_t^\nu + \partial_1 h(F_t, \kappa)\, (F_t + I_t^\nu)]\; \mathrm{d}W_t\,, \quad 0 \le t \le T\,,$$

is a martingale, so

$$\mathbb{E}\left[\int_0^T F_t\; [Y_t + Q_t^\nu + \partial_1 h(F_t, \kappa)\, (F_t + I_t^\nu)]\; \mathrm{d}W_t \right] = 0\,.$$

It follows that we may rewrite the performance criterion (16) as

$$(Y_0 + Q_0)\, F_0 + \left[\int_0^T \Big\{ \big(G_t + \sigma^2\, \partial_1 h(F_t, \kappa)\big)\; F_t^2 + (Y_t + Q_0)\, A_t\, F_t - \tfrac{\phi}{2}\, (Y_t + Q_0)^2 \Big\}\; \mathrm{d}t \right] + J[\nu]\,. \quad \text{(A1)}$$

Step 4. Because bounded processes are dense in $\mathcal{A}_2$, by continuity, (A1) holds for all $\nu \in \mathcal{A}_2$. □

Appendix C. Proof of Proposition 1

Consider the quadratic form $\mathcal{Q}$ and the linear functional $\mathcal{L}$ defined in Lemma 3. Define the symmetric bounded bilinear form $B : \mathcal{A}_2 \times \mathcal{A}_2 \to \mathbb{R}$ by

$$B(\nu, \zeta) = \frac{1}{4}\, (\mathcal{Q}(\nu + \zeta) - \mathcal{Q}(\nu - \zeta))\,.$$

Then

$$\begin{aligned}
B(\nu,\zeta) &= 2\,\eta\,\langle \nu,\zeta\rangle + \beta\,(\langle \mathfrak{L}\nu, \mathfrak{I}\zeta\rangle + \langle \mathfrak{L}\zeta, \mathfrak{I}\nu\rangle) - c\,(\langle \mathfrak{L}\nu,\zeta\rangle + \langle \mathfrak{L}\zeta,\nu\rangle) + \phi\,\langle \mathfrak{L}\nu, \mathfrak{L}\zeta\rangle \\
&= 2\,\eta\,\langle \nu,\zeta\rangle + \beta\,(\langle \mathfrak{I}^\top \mathfrak{L}\nu, \zeta\rangle + \langle \zeta, \mathfrak{L}^\top \mathfrak{I}\nu\rangle) - c\,(\langle \mathfrak{L}\nu,\zeta\rangle + \langle \zeta, \mathfrak{L}^\top\nu\rangle) + \phi\,\langle \mathfrak{L}^\top\mathfrak{L}\nu, \zeta\rangle \\
&= \langle (2\,\eta + \beta\,(\mathfrak{I}^\top\mathfrak{L} + \mathfrak{L}^\top\mathfrak{I}) - c\,(\mathfrak{L}+\mathfrak{L}^\top) + \phi\,\mathfrak{L}^\top\mathfrak{L})\nu, \zeta\rangle \\
&= \langle \Lambda\nu, \zeta\rangle
\end{aligned}$$

and

$$\mathcal{Q}(\nu) = B(\nu,\nu) = \langle \Lambda\nu,\nu\rangle\,.$$

For the linear functional $\mathcal{L}$, we have

$$\mathcal{L}(\nu) = \langle (\mathfrak{I}^\top (G\,F) + (c - \beta\,\mathfrak{I}^\top - \phi\,\mathfrak{L}^\top) + \mathfrak{L}^\top (A\,F), \nu\rangle = \langle b, \nu\rangle\,.$$

Therefore,

$$J[\nu] = -\frac{1}{2}\,\mathcal{Q}(\nu) + \mathcal{L}(\nu) = -\frac{1}{2}\,\langle \Lambda\nu,\nu\rangle + \langle b,\nu\rangle\,.$$

□

Appendix D. Proof of Proposition 2

Take $\nu \in \mathcal{A}_2$. Then

$$\begin{aligned}
\langle \Lambda\nu,\nu\rangle = \mathcal{Q}(\nu) &= 2\,\eta\,\|\nu\|^2 + 2\,\langle \mathfrak{L}\nu, \beta\,\mathfrak{I}\nu - c\,\nu\rangle + \phi\,\|\mathfrak{L}\nu\|^2 \\
&= 2\,\eta\,\|\nu\|^2 - 2\,c\,\langle \mathfrak{L}\nu,\nu\rangle + \phi\,\|\mathfrak{L}\nu\|^2 + 2\,\beta\langle \mathfrak{L}\nu, \mathfrak{I}\nu\rangle
\end{aligned}$$

By integration by parts, we have

$$\langle \mathfrak{L}\nu,\nu\rangle = \mathbb{E}\left[\int_0^T \int_0^t \nu_s\,\mathrm{d}s\,\nu_t\,\mathrm{d}t\right] = \mathbb{E}\left[\left(\int_0^T \nu_t\,\mathrm{d}t\right)^2 - \int_0^T\int_0^t \nu_s\,\mathrm{d}s\,\nu_t\,\mathrm{d}t\right]\,,$$

so

$$\langle \mathfrak{L}\nu,\nu\rangle = \frac{1}{2}\,\mathbb{E}\left[\left(\int_0^T \nu_t\,\mathrm{d}t\right)^2\right] \geq 0\,. \tag{A2}$$

Let

$$\tilde{\mathfrak{I}}_t \coloneqq \int_0^t e^{\beta\,(s-t)}\,\nu_s\,\mathrm{d}s\,.$$

The dynamics (12) implies

$$c\,\tilde{\mathfrak{I}}_t = I_t^\nu = c\int_0^t \nu_s\,\mathrm{d}s - \beta\int_0^t I_s^\nu\,\mathrm{d}s = c\,(\mathfrak{L}\nu)_t - \beta\int_0^t c\,\tilde{\mathfrak{I}}_s\,\mathrm{d}s\,,$$

so

$$c\,(\mathfrak{Q}\nu)_t = c\left(\tilde{\mathfrak{I}}_t + \beta\int_0^t \tilde{\mathfrak{I}}_s\,\mathrm{d}s\right).$$

Therefore,

$$\begin{aligned}
\langle \mathfrak{Q}\nu, \mathfrak{I}\nu\rangle = \mathbb{E}\left[\int_0^T (\mathfrak{Q}\nu)_t\,(\mathfrak{I}\nu)_t\,\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^T c\,(\mathfrak{Q}\nu)_t\,\tilde{\mathfrak{I}}_t\,\mathrm{d}t\right]\\
&= c\,\mathbb{E}\left[\int_0^T \left(\tilde{\mathfrak{I}}_t + \beta\int_0^t \tilde{\mathfrak{I}}_s\,\mathrm{d}s\right)\tilde{\mathfrak{I}}_t\,\mathrm{d}t\right]\\
&= c\,\mathbb{E}\left[\int_0^T \tilde{\mathfrak{I}}_t^2\,\mathrm{d}t + \beta\int_0^T \tilde{\mathfrak{I}}_t\int_0^t \tilde{\mathfrak{I}}_s\,\mathrm{d}s\,\mathrm{d}t\right]\\
&= c\left(\|\tilde{\mathfrak{I}}\|^2 + \beta\left\langle \mathfrak{Q}\tilde{\mathfrak{I}}, \tilde{\mathfrak{I}}\right\rangle\right)\\
&\geq 0
\end{aligned}$$

due to (A2). It follows that (recall Assumption 3)

$$\begin{aligned}
\langle \Lambda\nu, \nu\rangle &= 2\,\eta\,\|\nu\|^2 - 2\,c\,\langle \mathfrak{Q}\nu, \nu\rangle + \phi\,\|\mathfrak{Q}\nu\|^2 + 2\,\beta\langle \mathfrak{Q}\nu, \mathfrak{I}\nu\rangle\\
&\geq 2\,\eta\,\|\nu\|^2 - 2\,\sqrt{2\,\eta\,\phi}\,\langle \mathfrak{Q}\nu, \nu\rangle + \phi\,\|\mathfrak{Q}\nu\|^2\\
&= \left\|\sqrt{2\,\eta}\,\nu - \sqrt{\phi}\,\mathfrak{Q}\nu\right\|^2.
\end{aligned}$$

Consider the bounded linear operator $\mathfrak{V} : L^2[0,T] \to L^2[0,T]$ defined by

$$(\mathfrak{V}f)(t) = \sqrt{2\,\eta}f(t) - \sqrt{\phi}\int_0^t f(s)\,\mathrm{d}s\,,$$

whose inverse is also a bounded linear operator on $L^2[0,T]$ and is given by

$$(\mathfrak{V}^{-1}f)(t) = \frac{1}{\sqrt{2\,\eta}}\,f(t) + \frac{\sqrt{\phi}}{2\,\eta}\int_0^t e^{\sqrt{\frac{\phi}{2\,\eta}}\,(t-s)}\,f(s)\,\mathrm{d}s\,.$$

Since $\nu(\omega) \in L^2[0,T]$ for $\mathbb{P}$-a.e. ω, we have

$$\begin{aligned}
\|\nu\|^2 = \int_\Omega \|\nu(\omega)\|^2_{L^2[0,T]}\,\mathrm{d}\mathbb{P}(\omega) &= \int_\Omega \left\|\mathfrak{V}^{-1}\mathfrak{V}(\nu(\omega))\right\|^2_{L^2[0,T]}\,\mathrm{d}\mathbb{P}(\omega)\\
&\leq \int_\Omega \left\|\mathfrak{V}^{-1}\right\|^2_{\mathrm{op}}\,\|\mathfrak{V}(\nu(\omega))\|^2_{L^2[0,T]}\,\mathrm{d}\mathbb{P}(\omega)\\
&= \left\|\mathfrak{V}^{-1}\right\|^2_{\mathrm{op}}\left\|\sqrt{2\,\eta}\,\nu - \sqrt{\phi}\,\mathfrak{Q}\nu\right\|^2.
\end{aligned}$$

Therefore,

$$\langle \Lambda\nu, \nu\rangle \geq \left\|\mathfrak{V}^{-1}\right\|^{-2}_{\mathrm{op}}\,\|\nu\|^2\,.$$

so Λ is coercive. By Lax-Milgram lemma, Λ has an inverse, which is a bounded linear operator on $\mathcal{A}_2$.

Next, take $\nu, \zeta \in \mathcal{A}_2$ and $\rho \in (0,1)$. Then

$$\begin{aligned}
J[\rho\,\nu + (1-\rho)\,\zeta] &= -\frac{1}{2}\,\langle \Lambda(\rho\,\nu + (1-\rho)\,\zeta), \rho\,\nu + (1-\rho)\,\zeta\rangle + \langle b, \rho\,\nu + (1-\rho)\,\zeta\rangle \\
&= -\frac{1}{2}\,\big(\rho^2\,\langle \Lambda\nu, \nu\rangle + 2\,\rho\,(1-\rho)\,\langle \Lambda\nu, \zeta\rangle + (1-\rho)^2\,\langle \Lambda\zeta, \zeta\rangle\big) + \rho\,\langle b, \nu\rangle + (1-\rho)\,\langle b, \zeta\rangle \\
&= -\frac{1}{2}\,\big(\big(\rho^2 - \rho\big)\,\langle \Lambda\nu, \nu\rangle + 2\,\rho\,(1-\rho)\,\langle \Lambda\nu, \zeta\rangle + \big((1-\rho)^2 - (1-\rho)\big)\,\langle \Lambda\zeta, \zeta\rangle\big) \\
&\quad + \rho\,J[\nu] + (1-\rho)\,J[\zeta] \\
&= \frac{1}{2}\,\rho\,(1-\rho)\,\left(\langle \Lambda\nu, \nu\rangle - 2\,\langle \Lambda\nu, \zeta\rangle + \langle \Lambda\zeta, \zeta\rangle\right) + \rho\,J[\nu] + (1-\rho)\,J[\zeta] \\
&= \frac{1}{2}\,\rho\,(1-\rho)\,\langle \Lambda(\nu-\zeta), \nu-\zeta\rangle + \rho\,J[\nu] + (1-\rho)\,J[\zeta] \\
&= \frac{1}{2}\,\rho\,(1-\rho)\,\left\|\mathfrak{V}^{-1}\right\|_{\text{op}}^{-2}\,\|\nu-\zeta\|^2 + \rho\,J[\nu] + (1-\rho)\,J[\zeta] \\
&\geq \rho\,J[\nu] + (1-\rho)\,J[\zeta]\,,
\end{aligned}$$

with equality if and only if $\nu = \zeta$. Hence, J is strictly concave. □

Appendix E. Proof of Proposition 3

Take $\nu, \delta \in \mathcal{A}_2$ and $\epsilon > 0$. Then

$$\begin{aligned}
\frac{1}{\epsilon}\,(J[\nu + \epsilon\,\delta] - J[\nu]) &= \frac{1}{\epsilon}\,\left(-\frac{1}{2}\,\langle \Lambda(\nu + \epsilon\,\delta), \nu + \epsilon\,\delta\rangle + \langle b, \nu + \epsilon\,\delta\rangle + \frac{1}{2}\langle \Lambda\nu, \nu\rangle - \langle b, \nu\rangle\right) \\
&= \frac{1}{\epsilon}\,\left(-\frac{1}{2}\,\langle \Lambda\nu, \nu\rangle - \epsilon\,\langle \Lambda\nu, \delta\rangle - \frac{\epsilon^2}{2}\,\langle \Lambda\delta, \delta\rangle + \langle b, \epsilon\,\delta\rangle + \frac{1}{2}\langle \Lambda\nu, \nu\rangle\right) \\
&= -\langle \Lambda\nu, \delta\rangle - \frac{\epsilon}{2}\,\langle \Lambda\delta, \delta\rangle + \langle b, \delta\rangle
\end{aligned}$$

It follows that the Gâteaux derivative $\mathfrak{D}J[\nu]$ at $\nu \in \mathcal{A}_2$ is

$$\mathfrak{D}J[\nu](\delta) = \lim_{\epsilon\downarrow 0} \frac{J[\nu + \epsilon\delta] - J[\nu]}{\epsilon} = \langle -\Lambda\nu + b, \delta\rangle\,.$$

We identify $\mathfrak{D}J[\nu]$ with $-\Lambda\nu + b$. From (18) and (19), we get

$$\begin{aligned}
\mathfrak{D}J[\nu] = -\Lambda\nu + b = -2\,\eta\,\nu + c\,(Y + Q_0 + \mathfrak{Q}\nu) + \mathfrak{Q}^T(A\,F - \beta\,\mathfrak{I}\nu + c\,\nu - \phi\,(Y + Q_0 + \mathfrak{Q}\nu)) \\
+ \mathfrak{I}^\top(G\,F - \beta\,(Y + Q_0 + \mathfrak{Q}\nu))\,.
\end{aligned} \tag{A3}$$

Write

$$\begin{aligned}
\langle \mathfrak{Q}\nu, \zeta\rangle = \mathbb{E}\left[\int_0^T \int_0^t \nu_s\,\mathrm{d}s\,\zeta_t\,\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^T \nu_s \int_s^T \zeta_t\,\mathrm{d}t\,\mathrm{d}s\right] \\
&= \int_0^T \mathbb{E}\left[\nu_s \int_s^T \zeta_t\,\mathrm{d}t\right]\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \mathbb{E}\left[\mathbb{E}\left[\nu_s \int_s^T \zeta_t \,\mathrm{d}t \,\middle|\, \mathcal{F}_s\right]\right] \mathrm{d}s \\
&= \int_0^T \mathbb{E}\left[\nu_s\, \mathbb{E}\left[\int_s^T \zeta_t \,\mathrm{d}t \,\middle|\, \mathcal{F}_s\right]\right] \mathrm{d}s \\
&= \mathbb{E}\left[\int_0^T \nu_s\, \mathbb{E}\left[\int_s^T \zeta_t \,\mathrm{d}t \,\middle|\, \mathcal{F}_s\right] \mathrm{d}s\right],
\end{aligned}
$$

thus, $\mathfrak{Q}^\top$ is given by

$$
(\mathfrak{Q}^\top \zeta)_t = \mathbb{E}\left[\int_t^T \zeta_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\int_0^T \zeta_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] - \int_0^t \zeta_s \,\mathrm{d}s \,,
$$

where the in the last expression, the martingale term is càdlàg, so the entire process is càdlàg and thus progressively measurable. Similarly, since

$$
\langle \mathfrak{I}\nu, \zeta\rangle = \mathbb{E}\left[\int_0^T c \int_0^t e^{\beta\,(s-t)}\, \nu_s \,\mathrm{d}s\, \zeta_t \,\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T \nu_s\, c\, \mathbb{E}\left[\int_s^T e^{\beta\,(s-t)}\, \zeta_t \,\mathrm{d}t \,\middle|\, \mathcal{F}_s\right] \mathrm{d}s\right],
$$

$\mathfrak{I}^\top$ is given by

$$
(\mathfrak{I}^\top \zeta)_t = c\, \mathbb{E}\left[\int_t^T e^{\beta\,(t-s)}\, \zeta_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right].
$$

It follows from (A3) that

$$
\begin{aligned}
\mathfrak{D}J[\nu]_t = -2\,\eta\,\nu_t + c\,(Y_t + Q_t^\nu) &+ \mathbb{E}\left[\int_t^T (A_s\,F_s + c\,\nu_s - \beta\, I_s^\nu - \phi\,(Y_s + Q_s^\nu))\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&+ c\, e^{t\,\beta}\, \mathbb{E}\left[\int_t^T e^{-s\,\beta}\,(G_s\,F_s - \beta\,(Y_s + Q_s^\nu))\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right].
\end{aligned}
$$

□

Appendix F. Proof of Theorem 1

Suppose $\mathfrak{D}J[\nu^\star] = 0$ for some $\nu^\star \in \mathcal{A}_2$. Then by Proposition 3 we have

$$
\begin{aligned}
2\,\eta\,\nu_t^\star &= \mathbb{E}\left[c\,\left(Y_t + Q_t^{\nu^\star}\right) + \int_t^T \left(A_s\,F_s + c\,\nu_s^\star - \beta\, I_s^{\nu^\star} - \phi\left(Y_s + Q_s^{\nu^\star}\right)\right) \mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&\quad + c\, e^{t\,\beta}\, \mathbb{E}\left[\int_t^T e^{-s\,\beta} \left(G_s\,F_s - \beta\left(Y_s + Q_s^{\nu^\star}\right)\right) \mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}\left[c\,\left(Y_T + Q_T^{\nu^\star}\right) + \int_t^T \left((A_s - c\,G_s)\,F_s - \beta\, I_s^{\nu^\star} - \phi\left(Y_s + Q_s^{\nu^\star}\right)\right) \mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&\quad + c\, e^{t\,\beta}\, \mathbb{E}\left[\int_t^T e^{-s\,\beta} \left(G_s\,F_s - \beta\left(Y_s + Q_s^{\nu^\star}\right)\right) \mathrm{d}s \,\middle|\, \mathcal{F}_t\right] - c\,\sigma\, \mathbb{E}\left[\int_t^T \partial_1 h(F_s, \kappa)\, F_s \,\mathrm{d}W_s \,\middle|\, \mathcal{F}_t\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}\left[c\left(Y_T + Q_T^{\nu^\star}\right) + \int_0^T \left((A_s - c\,G_s)\, F_s - \beta\, I_s^{\nu^\star} - \phi\left(Y_s + Q_s^{\nu^\star}\right)\right) \mathrm{d}s \,\middle|\, \mathcal{F}_t \right] \\
&- \int_0^t \left((A_s - c\,G_s)\, F_s - \beta\, I_s^{\nu^\star} - \phi\left(Y_s + Q_s^{\nu^\star}\right)\right) \mathrm{d}s + c\, e^{t\,\beta}\, \mathbb{E}\left[\int_0^T e^{-s\,\beta} \left(G_s\, F_s - \beta \left(Y_s + Q_s^{\nu^\star}\right)\right) \mathrm{d}s \,\middle|\, \mathcal{F}_t \right] \\
&- c\, e^{t\,\beta} \int_0^t e^{-s\,\beta} \left(G_s\, F_s - \beta \left(Y_s + Q_s^{\nu^\star}\right)\right) \mathrm{d}s - c\,\sigma\, \mathbb{E}\left[\int_t^T \partial_1 h(F_s, \kappa)\, F_s\, \mathrm{d}W_s \,\middle|\, \mathcal{F}_t \right].
\end{aligned}
$$

Since

$$
\mathbb{E}\left[\int_0^T \left|\partial_1 h(F_t, \kappa)\right|^2\, F_t^2\, \mathrm{d}t \right] \lesssim \mathbb{E}\left[\int_0^T \left(F_t^{2\,q_\kappa + 2} + F_t^{2\,p_\kappa + 2} \right) \mathrm{d}t \right] < \infty\,,
$$

the process

$$
\int_0^t \partial_1 h(F_s, \kappa)\, F_s\, \mathrm{d}W_s\,, \quad 0 \le t \le T\,, \tag{A4}
$$

is a martingale, so

$$
\mathbb{E}\left[\int_t^T \partial_1 h(F_s, \kappa)\, F_s\, \mathrm{d}W_s \,\middle|\, \mathcal{F}_t \right] = 0\,.
$$

Define process $\tilde{N}$ by

$$
\tilde{N}_t = \mathbb{E}\left[\int_0^T e^{-s\,\beta} \left(G_s\, F_s - \beta \left(Y_s + Q_s^{\nu^\star}\right)\right) \mathrm{d}s \,\middle|\, \mathcal{F}_t \right].
$$

Then $\tilde{N}$ is a martingale with

$$
\begin{aligned}
\mathbb{E}\left[|\tilde{N}_T|^2\right] &\le \mathbb{E}\left[\left| \int_0^T e^{-s\,\beta} \left(G_s\, F_s - \beta \left(Y_s + Q_s^{\nu^\star}\right)\right) \mathrm{d}s \right|^2 \right] \\
&\lesssim \mathbb{E}\left[\int_0^T |G_s\, F_s|^2\, \mathrm{d}s \right] + \mathbb{E}\left[\int_0^T |Y_s|^2\, \mathrm{d}s \right] + \mathbb{E}\left[\int_0^T \left|Q_s^{\nu^\star}\right|^2 \mathrm{d}s \right] \\
&\lesssim \mathbb{E}\left[\int_0^T |G_s|^q\, \mathrm{d}s \right]^{2/q} \mathbb{E}\left[\int_0^T |F_s|^r\, \mathrm{d}s \right]^{2/r} + \mathbb{E}\left[\int_0^T |Y_s|^2\, \mathrm{d}s \right] + \mathbb{E}\left[\int_0^T \left|Q_s^{\nu^\star}\right|^2 \mathrm{d}s \right] \\
&< \infty
\end{aligned}
$$

for some $q \in (2, p)$ and $r > 2$ such that $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$ due to Lemma 1. Define process Z by

$$
Z_t = e^{t\,\beta} \left(\tilde{N}_t - \int_0^t e^{-s\,\beta} \left(G_s\, F_s - \beta \left(Y_s + Q_s^{\nu^\star}\right)\right) \mathrm{d}s \right).
$$

Then $Z_T = 0$, and generalized Itô's formula (note $\tilde{N}$ is càdlàg but not necessarily continuous) implies

$$
-Z_t = \int_t^T \beta\, e^{s\,\beta} \left(\tilde{N}_{s-} - \int_0^s e^{-u\,\beta} \left(G_u\, F_u - \beta \left(Y_u + Q_u^{\nu^\star}\right)\right) \mathrm{d}u \right) \mathrm{d}s + \int_t^T e^{s\,\beta}\, \mathrm{d}\tilde{N}_s
$$

$$
\begin{aligned}
&- \int_t^T \left(G_s\, F_s - \beta \left(Y_s + Q_s^{\nu^\star} \right) \right) \mathrm{d}s + \sum_{t < s \le T} \left[e^{s\,\beta}\, \tilde{N}_s - e^{s\,\beta}\, \tilde{N}_{s-} - e^{s\,\beta}\, \Delta \tilde{N}_s \right] \\
&= \int_t^T \beta\, e^{s\,\beta} \left(\tilde{N}_s - \int_0^s e^{-u\,\beta} \left(G_u\, F_u - \beta \left(Y_u + Q_u^{\nu^\star} \right) \right) \mathrm{d}u \right) \mathrm{d}s + \int_t^T e^{s\,\beta}\, \mathrm{d}\tilde{N}_s \\
&\quad - \int_t^T \left(G_s\, F_s - \beta \left(Y_s + Q_s^{\nu^\star} \right) \right) \mathrm{d}s \\
&= \int_t^T \left(\beta \left(Z_s + Y_s + Q_s^{\nu^\star} \right) - G_s\, F_s \right) \mathrm{d}s + \int_t^T e^{s\,\beta}\, \mathrm{d}\tilde{N}_s \,,
\end{aligned}
$$

where the second equality is because a càdlàg path has at most countably many jumps, which form a Lebesgue measure zero set. Define process N by

$$
N_t = \int_0^t e^{s\,\beta}\, \mathrm{d}\tilde{N}_s \,, \quad 0 \le t \le T \,.
$$

Since

$$
\mathbb{E}\left[\int_0^T e^{2\,s\,\eta}\, \mathrm{d}\langle \tilde{N} \rangle_s \right] \le e^{2\,T\,\eta}\, \mathbb{E}\left[\langle \tilde{N} \rangle_T \right] \le e^{2\,T\,\eta}\, \mathbb{E}\left[|\tilde{N}_T|^2 \right] < \infty \,,
$$

N is a martingale with $N_T \in L^2(\Omega)$. Moreover, the process M, defined by

$$
M_t = \mathbb{E}\left[c \left(Y_T + Q_T^{\nu^\star} \right) + \int_0^T \left((A_s - c\, G_s)\; F_s - \beta\, I_s^{\nu^\star} - \phi \left(Y_s + Q_s^{\nu^\star} \right) \right) \mathrm{d}s \,\middle|\, \mathcal{F}_t \right] + c\, N_t \,,
$$

is also a martingale with $M_T \in L^2(\Omega)$. Combining everything gives

$$
\begin{aligned}
2\,\eta\, \nu_s^\star &= M_t - c\, N_t - \int_0^t \left((A_s - c\, G_s)\; F_s - \beta\, I_s^{\nu^\star} - \phi \left(Y_s + Q_s^{\nu^\star} \right) \right) \mathrm{d}s + c\, Z_t \\
&= M_t - c\, N_t - \int_0^t \left((A_s - c\, G_s)\; F_s - \beta\, I_s^{\nu^\star} - \phi \left(Y_s + Q_s^{\nu^\star} \right) \right) \mathrm{d}s \\
&\quad - c \int_t^T \left(\beta \left(Z_s + Y_s + Q_s^{\nu^\star} \right) - G_s\, F_s \right) \mathrm{d}s - c\,(N_T - N_t) \\
&= M_t - \int_0^T \left((A_s - c\, G_s)\; F_s - \beta\, I_s^{\nu^\star} - \phi \left(Y_s + Q_s^{\nu^\star} \right) \right) \mathrm{d}s \\
&\quad + \int_t^T \left(A_s\, F_s - \beta\, I_s^{\nu^\star} - (\phi + c\,\beta) \left(Y_s + Q_s^{\nu^\star} \right) - c\,\beta\, Z_s \right) \mathrm{d}s - cN_T \\
&= c \left(Y_T + Q_T^{\nu^\star} \right) - \int_t^T \left(-A_s\, F_s - \beta\, I_s^{\nu^\star} + (\phi + c\,\beta) \left(Y_s + Q_s^{\nu^\star} \right) + c\,\beta\, Z_s \right) \mathrm{d}s - (M_T - M_t) \,.
\end{aligned}
$$

Thus $\nu^\star$ satisfies the FBSDE (21).

Conversely, assume $\nu^\star \in \mathcal{A}_2$ satisfies the FBSDE (21) for some martingales M and N such that $M_T, N_T \in L^2(\Omega)$. By integrating $\nu^\star$ and Z and using the terminal conditions, we may write

$$
2\,\eta\, \nu_t^\star = c\,(Y_T + Q_T) + \int_t^T (A_s\, F_s - \beta\, I_s - (\phi + c\,\beta)\,(Y_s + Q_s) - c\,\beta\, Z_s)\; \mathrm{d}s - M_T + M_t
$$

and

$$Z_t = \int_t^T \left(-\beta\,(Z_s + Y_s + Q_s) + G_s\,F_s\right)\,\mathrm{d}s - N_T + N_t$$

Combining above two identities as well as the dynamics of Y and Q gives

$$\begin{aligned}
2\,\eta\,\nu_t^\star &= c\,(Y_t + Q_t) + \int_t^T c\,G_s\,F_s\,\mathrm{d}s + \int_t^T c\,\sigma\,\partial_1 h(F_s,\kappa)\,F_s\,\mathrm{d}W_s + \int_t^T c\,\nu_s^\star\,\mathrm{d}s \\
&\quad + \int_t^T \left(A_s\,F_s - \beta\,I_s - (\phi + c\,\beta)\,(Y_s + Q_s)\right)\,\mathrm{d}s - M_T + M_t \\
&\quad + c\,Z_t + \int_t^T \left(c\,\beta\,(Y_s + Q_s) - c\,G_s\,F_s\right)\,\mathrm{d}s + c\,N_T - c\,N_t \\
&= c\,(Y_t + Q_t) + \int_t^T \left(A_s\,F_s + c\,\nu_s^\star - \beta\,I_s - \phi\,(Y_s + Q_s)\right)\,\mathrm{d}s + c\,Z_t \\
&\quad + \int_t^T c\,\sigma\,\partial_1 h(F_s,\kappa)\,F_s\,\mathrm{d}W_s - M_T + M_t + c\,N_T - c\,N_t\,. \qquad \text{(A5)}
\end{aligned}$$

Recall that the process in (A4) is a martingale, so taking conditional expectation on above equation gives

$$2\,\eta\,\nu_t^\star = c\,(Y_t + Q_t) + \mathbb{E}\left[\int_t^T \left(A_s\,F_s + c\,\nu_s^\star - \beta\,I_s - \phi\,(Y_s + Q_s)\right)\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right] + c\,\mathbb{E}[Z_t\,|\,\mathcal{F}_t]$$

To solve for Z, we use generalized Itô's formula and the dynamics and terminal condition of Z to write

$$\begin{aligned}
Z_t = e^{t\,\beta}\,e^{-t\,\beta}\,Z_t &= e^{t\,\beta}\left(\int_t^T \beta\,e^{-s\,\beta}\,Z_s\,\mathrm{d}s - \int_t^T e^{-s\,\beta}\,\left(\beta\,(Z_s + Y_s + Q_s) - G_s\,F_s\right)\,\mathrm{d}s - \int_t^T e^{-s\,\beta}\,\mathrm{d}N_s\right) \\
&= -e^{t\,\beta}\left(\int_t^T e^{-s\,\beta}\,\left(\beta\,(Y_s + Q_s) - G_s\,F_s\right)\,\mathrm{d}s + \int_t^T e^{-s\,\beta}\,\mathrm{d}N_s\right)\,.
\end{aligned}$$

Since

$$\mathbb{E}\left[\int_0^T e^{-2\,s\,\eta}\,\mathrm{d}\langle N\rangle_s\right] \le \mathbb{E}\left[\langle N\rangle_T\right] \le \mathbb{E}\left[|N_T|^2\right] < \infty\,,$$

the process

$$\int_0^t e^{-s\,\beta}\,\mathrm{d}N_s\,,\quad 0 \le t \le T\,,$$

is a martingale. Therefore,

$$\mathbb{E}[Z_t\,|\,\mathcal{F}_t] = e^{t\,\beta}\,\mathbb{E}\left[\int_t^T e^{-s\,\beta}\,\left(G_s\,F_s - \beta\,(Y_s + Q_s)\right)\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right]\,.$$

Plugging this into (A5) gives

$$
\begin{aligned}
2\,\eta\,\nu_t^\star = c\,(Y_t + Q_t) + \mathbb{E}\left[\int_t^T (A_s\,F_s + c\,\nu_s^\star - \beta\,I_s - \phi\,(Y_s + Q_s))\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
+ c\,e^{t\,\beta}\,\mathbb{E}\left[\int_t^T e^{-s\,\beta}\,(G_s\,F_s - \beta\,(Y_s + Q_s))\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right],
\end{aligned}
$$

that is, $\mathfrak{D}J[\nu^\star]_t = 0$.

Appendix G. Proof of Proposition 4

First, we show we may construct a solution of the FBSDE from a solution of the DRE. Suppose P is a solution to the DRE (22) and the processes ℓ, Φ, Ψ are defined as stated in the proposition. Let us differentiate these processes. For Φ, we have

$$
\begin{aligned}
\mathrm{d}\Phi_t &= (B_{12}\,P(t) + B_{11})\;\Phi_t\,\mathrm{d}t + e^{\int_0^t (B_{12}\,P(u) + B_{11})\,\mathrm{d}u}\,e^{-\int_0^t (B_{12}\,P(u) + B_{11})\,\mathrm{d}u}\,B_{12}\,\ell_t\,\mathrm{d}t \\
&= (B_{12}\;(P(t)\,\Phi_t + \ell_t) + B_{11}\,\Phi_t)\;\mathrm{d}t \\
&= (B_{12}\,\Psi_t + B_{11}\,\Phi_t)\;\mathrm{d}t\,.
\end{aligned}
$$

For ℓ, we have

$$
\begin{aligned}
\ell_t &= e^{-\int_0^t (P(u)\,B_{12} - B_{22})\,\mathrm{d}u}\,\mathbb{E}\left[L - \int_t^T e^{\int_0^s (P(u)\,B_{12} - B_{22})\,\mathrm{d}u}\,b_s\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&= e^{-\int_0^t (P(u)\,B_{12} - B_{22})\,\mathrm{d}u}\left(\mathbb{E}\left[L - \int_0^T e^{\int_0^s (P(u)\,B_{12} - B_{22})\,\mathrm{d}u}\,b_s\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] + \int_0^t e^{\int_0^s (P(u)\,B_{12} - B_{22})\,\mathrm{d}u}\,b_s\,\mathrm{d}s\right).
\end{aligned}
$$

Let

$$
\tilde{\mathcal{M}}_t = \mathbb{E}\left[L - \int_0^T e^{\int_0^s (P(u)\,B_{12} - B_{22})\,\mathrm{d}u}\,b_s\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right],
$$

then $\tilde{M}$ is an $\mathbb{R}^2$-valued martingale. By Lemma 1, we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T |b_t|^2\,\mathrm{d}t\right] \\
&\lesssim \mathbb{E}\left[\int_0^T \left(|A_t\,F_t|^2 + |Y_t|^2 + |G_t\,F_t|^2\right)\,\mathrm{d}t\right] \\
&\leq \mathbb{E}\left[\int_0^T |A_t|^p\,\mathrm{d}t\right]^{\frac{2}{p}}\,\mathbb{E}\left[\int_0^T |F_t|^{\frac{2\,p}{p-2}}\,\mathrm{d}t\right]^{\frac{p-2}{p}} + \mathbb{E}\left[\int_0^T |F_t|^2\,\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T |G_t|^q\,\mathrm{d}t\right]^{\frac{2}{q}}\,\mathbb{E}\left[\int_0^T |F_t|^{\frac{2\,q}{q-2}}\,\mathrm{d}t\right]^{\frac{q-2}{q}} \\
&< \infty
\end{aligned}
\tag{A6}
$$

for some $q \in (2, p)$, and thus

$$\mathbb{E}\left[\left|\tilde{\mathcal{M}}_T\right|^2\right] \le \mathbb{E}\left[\left|L - \int_0^T e^{\int_0^s (P(u)\,B_{12} - B_{22})\,\mathrm{d}u}\, b_s\,\mathrm{d}s\right|^2\right] \lesssim \mathbb{E}[Y_T^2] + \mathbb{E}\left[\int_0^T |b_s|^2\,\mathrm{d}s\right] < \infty\,. \tag{A7}$$

By generalized Itô's formula,

$$\begin{aligned}
\mathrm{d}\ell_t &= -\left(P(t)\,B_{12} - B_{22}\right)\,\ell_t\,\mathrm{d}t + e^{-\int_0^t (P(u)\,B_{12} - B_{22})\,\mathrm{d}u}\left(\mathrm{d}\tilde{\mathcal{M}}_t + e^{\int_0^t (P(u)\,B_{12} - B_{22})\,\mathrm{d}u}\, b_t\,\mathrm{d}t\right) \\
&= \left(\left(-P(t)\,B_{12} + B_{22}\right)\,\ell_t + b_t\right)\,\mathrm{d}t + e^{-\int_0^t (P(u)\,B_{12} - B_{22})\,\mathrm{d}u}\,\mathrm{d}\tilde{\mathcal{M}}_t\,.
\end{aligned}$$

Let

$$\mathcal{M}_t = \int_0^t e^{-\int_0^s (P(u)\,B_{12} - B_{22})\,\mathrm{d}u}\,\mathrm{d}\tilde{\mathcal{M}}_s\,.$$

Since the integrand is deterministic and differentiable and because of (A7), $\mathcal{M}$ is an $\mathbb{R}^2$-valued martingale with $\mathbb{E}[|\mathcal{M}_T|]^2 < \infty$. For Ψ, we have

$$\begin{aligned}
\mathrm{d}\Psi_t &= P'(t)\,\Phi_t\,\mathrm{d}t + P(t)\,\mathrm{d}\Phi_t + \mathrm{d}\ell_t \\
&= P'(t)\,\Phi_t\,\mathrm{d}t + P(t)\left(B_{11}\,\Phi_t + B_{12}\left(P(t)\,\Phi_t + \ell_t\right)\right)\mathrm{d}t + \mathrm{d}\ell_t \\
&= \left(P'(t) + P(t)\,B_{11} + P(t)\,B_{12}\,P(t)\right)\Phi_t\,\mathrm{d}t + P(t)\,B_{12}\,\ell_t\,\mathrm{d}t + \mathrm{d}\ell_t \\
&= \left(B_{21} + B_{22}\,P(t)\right)\Phi_t\,\mathrm{d}t + P(t)\,B_{12}\,\ell_t\,\mathrm{d}t + \left(\left(-P(t)\,B_{12} + B_{22}\right)\ell_t + b_t\right)\mathrm{d}t + \mathrm{d}\mathcal{M}_t \\
&= \left(B_{21}\,\Phi_t + B_{22}\left(P(t)\,\Phi_t + \ell_t\right) + b_t\right)\mathrm{d}t + \mathrm{d}\mathcal{M}_t \\
&= \left(B_{21}\,\Phi_t + B_{22}\,\Psi_t + b_t\right)\mathrm{d}t + \mathrm{d}\mathcal{M}_t\,.
\end{aligned}$$

Thus we obtain the FBSDE

$$\begin{cases}
\mathrm{d}\Phi_t &= \left(B_{11}\,\Phi_t + B_{12}\,\Psi_t\right)\,\mathrm{d}t\,, \qquad \Phi_0 = K \\
\mathrm{d}\Psi_t &= \left(B_{21}\,\Phi_t + B_{22}\,\Psi_t + b_t\right)\,\mathrm{d}t + \mathrm{d}\mathcal{M}_t\,, \quad \Psi_T = G\,\Phi_T + L
\end{cases},$$

which is precisely FBSDE (21) written in vectorial form, provided we identify

$$\Psi_t = \begin{pmatrix} \nu_t^\star \\ Z_t \end{pmatrix}, \quad \Phi_t = \begin{pmatrix} I_t \\ Q_t \end{pmatrix}, \quad \mathcal{M}_t = \begin{pmatrix} \frac{1}{2\,\eta}\,M_t \\ N_t \end{pmatrix}.$$

Moreover, due to (A6) and (A7), we obtain the three inequalities

$$\begin{aligned}
\mathbb{E}\left[\int_0^T |\ell_t|^2\,\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^T \left|e^{-\int_0^t (P(u)\,B_{12} - B_{22})\,\mathrm{d}u}\left(\tilde{\mathcal{M}}_t + \int_0^t e^{\int_0^s (P(u)\,B_{12} - B_{22})\,\mathrm{d}u}\, b_s\,\mathrm{d}s\right)\right|^2\,\mathrm{d}t\right] \\
&\lesssim \mathbb{E}\left[\int_0^T \left(\left|\tilde{\mathcal{M}}_t\right|^2 + \int_0^t |b_s|^2\,\mathrm{d}s\right)\,\mathrm{d}t\right]
\end{aligned}$$

$$\lesssim \mathbb{E}\left[\left|\tilde{\mathcal{M}}_T\right|^2\right] + \mathbb{E}\left[\int_0^T |b_t|^2\,\mathrm{d}t\right]$$

$$< \infty\,,$$

$$\mathbb{E}\left[\int_0^T |\Phi_t|^2\,\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T \left|e^{\int_0^t (B_{12}\,P(u)+B_{11})\,\mathrm{d}u}\left(K + \int_0^t e^{-\int_0^s (B_{12}\,P(u)+B_{11})\,\mathrm{d}u}\,B_{12}\,\ell_s\,\mathrm{d}s\right)\right|^2\,\mathrm{d}t\right]$$

$$\lesssim Q_0^2 + \mathbb{E}\left[\int_0^T\int_0^t |\ell_s|^2\,\mathrm{d}s\,\mathrm{d}t\right]$$

$$\lesssim Q_0^2 + \mathbb{E}\left[\int_0^T |\ell_t|^2\,\mathrm{d}t\right]$$

$$< \infty\,,$$

and

$$\mathbb{E}\left[\int_0^T |\Psi_t|^2\,\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T |P(t)\,\Phi_t + \ell_t|^2\,\mathrm{d}t\right] \lesssim \mathbb{E}\left[\int_0^T |\Phi_t|^2\,\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T |\ell_t|^2\,\mathrm{d}t\right] < \infty\,,$$

which implies $\nu^\star \in \mathcal{A}_2$.

Next, we show the DRE (22) admits a unique solution under Assumption 3, that is, $c^2 < 2\,\eta\,\phi$. Here we only consider the case where $c > 0$. The $c = 0$ case is addressed in Proposition 5, where we derive an explicit solution of (22). Let

$$z = -\frac{1}{2}\left(\frac{c^2}{2\,\beta} + \sqrt{\frac{\phi\,c^2\,\eta}{2\,\beta^2}}\right) < 0$$

and

$$w = \frac{2\,\beta\,z^2}{c\,\eta}\,.$$

Since

$$\sqrt{\frac{\phi\,c^2\,\eta}{2\,\beta^2}} > \sqrt{\frac{c^4}{4\,\beta^2}} = \frac{c^2}{2\,\beta}\,,$$

we have

$$-\sqrt{\frac{\phi\,c^2\,\eta}{2\,\beta^2}} < z < -\frac{c^2}{2\,\beta}\,. \tag{A8}$$

Let

$$C = \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix}\,,\quad D = \begin{pmatrix} 0 & \frac{z}{2\,\eta} \\ z & -\frac{c\,z}{2\,\eta} \end{pmatrix}\,,$$

and

$$\mathcal{L} = \begin{pmatrix} C\,B_{11} + D\,B_{21} & C\,B_{12} + B_{11}^\top\,D + D\,B_{22} \\ 0 & B_{12}^\top\,D \end{pmatrix}\,.$$

Consider

$$\mathcal{L} + \mathcal{L}^\top = \begin{pmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} \\ \mathcal{K}_{12}^\top & \mathcal{K}_{22} \end{pmatrix}\,,$$

where

$$\mathcal{K}_{11} = C\,B_{11} + (C\,B_{11})^\top + D\,B_{21} + (D\,B_{21})^\top = \begin{pmatrix} -2\,\beta & \frac{\beta\,z}{k} \\ \frac{\beta\,z}{k} & \frac{\phi\,z}{k} \end{pmatrix},$$

$$\mathcal{K}_{12} = C\,B_{12} + B_{11}^\top\,D + D\,B_{22} = \begin{pmatrix} c & 0 \\ w & 0 \end{pmatrix},$$

$$\mathcal{K}_{22} = B_{12}^\top\,D + D^\top\,B_{12} = \begin{pmatrix} 2\,z & 0 \\ 0 & 0 \end{pmatrix}.$$

We have

$$\mathcal{K}_{22} \preceq 0 \tag{A9}$$

and

$$(I - \mathcal{K}_{22}\,\mathcal{K}_{22}^\dagger)\,\mathcal{K}_{12}^\top = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & w \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{A10}$$

Also, consider

$$\mathcal{K}_{11} - \mathcal{K}_{12}\,\mathcal{K}_{22}^\dagger\,\mathcal{K}_{12}^\top = \begin{pmatrix} -2\,\beta & \frac{\beta\,z}{k} \\ \frac{\beta\,z}{k} & \frac{\phi\,z}{k} \end{pmatrix} - \begin{pmatrix} c & 0 \\ w & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2\,z} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & w \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -2\,\beta - \frac{c^2}{2\,z} & 0 \\ 0 & \frac{\phi\,z}{k} - \frac{w^2}{2\,z} \end{pmatrix}.$$

Due to (A8), we have

$$-2\,\beta - \frac{c^2}{2\,z} < -2\,\beta + \frac{c^2}{2} \cdot \frac{2\,\beta}{c^2} = -\beta < 0$$

and

$$\frac{\phi\,z}{k} - \frac{w^2}{2\,z} = \frac{\phi\,z}{k} - \frac{2\,\beta^2\,z^3}{c^2\,\eta^2} = \frac{z}{k}\left(\phi - \frac{2\,\beta^2\,z^2}{c^2\,\eta}\right) < \frac{z}{k}\left(\phi - \frac{2\,\beta^2}{c^2\,\eta} \cdot \frac{\phi\,c^2\,\eta}{2\,\beta^2}\right) = 0,$$

so

$$\mathcal{K}_{11} - \mathcal{K}_{12}\,\mathcal{K}_{22}^\dagger\,\mathcal{K}_{12}^\top \prec 0\,. \tag{A11}$$

Combining (A9), (A10), and (A11), we conclude

$$\mathcal{L} + \mathcal{L}^\top \preceq 0\,.$$

Moreover,

$$C + D\,G + G^\top\,D^\top = \begin{pmatrix} 1 & 0 \\ 0 & w + \frac{c\,z}{k} \end{pmatrix} \succ 0\,,$$

since (A8) implies

$$w + \frac{c\,z}{k} = \frac{2\,\beta\,z^2}{c\,\eta} + \frac{c\,z}{k} = z\left(\frac{2\,\beta\,z}{c\,\eta} + \frac{c}{k}\right) > z\left(-\frac{2\,\beta}{c\,\eta} \cdot \frac{c^2}{2\,\beta} + \frac{c}{k}\right) = 0\,.$$

By Theorem 2.3 in Freiling et al. (2000), DRE (22) has a unique solution.

□

Appendix H. Proof of Proposition 5

The LP's optimisation problem reduces to solving the following simplified FBSDE:

$$\begin{cases} \mathrm{d}\nu_t &= \left(\frac{\phi}{2\,\eta}\,Q_t + \frac{-A_t\,F_t+\phi\,Y_t}{2\,\eta}\right)\mathrm{d}t + \frac{1}{2\,\eta}\,\mathrm{d}M_t, \quad \nu_T \quad = 0 \\ \mathrm{d}Q_t &= \nu_t\,\mathrm{d}t, \end{cases}$$

and the ansatz $\nu_t = P(t)\,Q_t + \ell_t$ gives the equations

$$P'(t) = -P(t)^2 + \frac{\phi}{2\,\eta}, \quad P(T) = 0 \tag{A12}$$

and

$$\mathrm{d}\ell_t = \left(-P(t)\,\ell_t + \frac{-A_t\,F_t+\phi\,Y_t}{2\,\eta}\right)\mathrm{d}t + \frac{1}{2\,\eta}\,\mathrm{d}M_t, \quad \ell_T = 0 \tag{A13}$$

The solution of (A12) is

$$P(t) = \sqrt{\frac{\phi}{2\,\eta}}\tanh\left(\sqrt{\frac{\phi}{2\,\eta}}(t-T)\right).$$

To solve (A13), we define

$$\tilde{P}(s,t) := e^{\int_s^t P(u)\,\mathrm{d}u} = \frac{\cosh\left(\sqrt{\frac{\phi}{2\,\eta}}(t-T)\right)}{\cosh\left(\sqrt{\frac{\phi}{2\,\eta}}(s-T)\right)}$$

and use generalized Itô's formula to write

$$\begin{aligned}
\tilde{P}(0,t)\,\ell_t &= -\int_t^T \tilde{P}(0,s)\,P(s)\,\ell_s\,\mathrm{d}s - \int_t^T \tilde{P}(0,s)\,\mathrm{d}\ell_s \\
&= -\int_t^T \tilde{P}(0,s)\,P(s)\,\ell_s\,\mathrm{d}s - \int_t^T \tilde{P}(0,s)\,\left(-P(s)\,\ell_s + \frac{-A_s\,F_s+\phi\,Y_s}{2\,\eta}\right)\mathrm{d}s - \frac{1}{2\,\eta}\int_t^T \tilde{P}(0,s)\,\mathrm{d}M_s \\
&= \frac{1}{2\,\eta}\int_t^T \tilde{P}(0,s)\,(A_s\,F_s - \phi\,Y_s)\,\mathrm{d}s - \frac{1}{2\,\eta}\int_t^T \tilde{P}(0,s)\,\mathrm{d}M_s,
\end{aligned}$$

therefore,

$$\ell_t = \frac{1}{2\,\eta}\,\mathbb{E}\left[\int_t^T \tilde{P}(t,s)\,(A_s\,F_s - \phi\,Y_s)\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right].$$

Similarly, Q is obtained by solving the equation

$$\mathrm{d}Q_t = (P(t)\,Q_t + \ell_t)\,\mathrm{d}t,$$

whose solution is

$$Q_t = Q_0\,\tilde{P}(0,t) + \int_0^t \tilde{P}(s,t)\,\ell_s\,\mathrm{d}s. \tag{A14}$$

Finally,

$$\begin{aligned}\nu_t &= P(t)\,Q_t + \ell_t \\ &= P(t)\left(Q_0\,\tilde{P}(0,t) + \int_0^t \tilde{P}(s,t)\,\ell_s\,\mathrm{d}s\right) + \ell_t\end{aligned}$$

□

Appendix I. Proof of Proposition 6

Let $\kappa > 0$. By Lemma 3, The quantity (26) can be written as

$$J[\nu^\star] + H + \tilde{H}\,,$$

where

$$H = \mathbb{E}\left[\int_0^T \left\{\left(G_t + \sigma^2\,\partial_1 h(F_t,\kappa)\right)\,F_t^2 + A_t\,F_t\,(Y_t - Y_0) - \tfrac{\phi}{2}\,(Y_t - Y_0)^2\right\}\,\mathrm{d}t\right]$$

and

$$\tilde{H} \coloneqq \mathbb{E}\left[\int_0^T \Pi(F_t,\kappa)\,\mathrm{d}t + X_T\right]\,.$$

Since H and $J[\nu^\star]$ are well-defined, it remains to show $\tilde{H}$ is well-defined. Recall that $h(\cdot,\kappa)$ is the inverse of $-\partial_1\varphi(\cdot,\kappa)$, so

$$-\partial_{11}\varphi(h(x,\kappa),\kappa) = \frac{1}{\partial_1 h(x,\kappa)}\,,\quad \forall x > 0\,. \tag{A15}$$

By Itô's formula and (13), we have

$$\begin{aligned}X_T = \varphi(Y_T,\kappa) &= \varphi(Y_0,\kappa) + \int_0^T \partial_1\varphi(Y_t,\kappa)\,\mathrm{d}Y_t + \frac{1}{2}\int_0^T \partial_{11}\varphi(Y_t,\kappa)\,\mathrm{d}\langle Y\rangle_t \\ &= \varphi(Y_0,\kappa) - \int_0^T F_t\,\mathrm{d}Y_t - \frac{1}{2}\int_0^T \frac{1}{\partial_1 h(F_t,\kappa)}\,\mathrm{d}\langle Y\rangle_t \\ &= \varphi(h(F_0,\kappa),\kappa) - \int_0^T \left(G_t + \frac{\sigma^2}{2}\,\partial_1 h(F_t,\kappa)\right)\,F_t^2\,\mathrm{d}t - \sigma\int_0^T \partial_1 h(F_t,\kappa)\,F_t^2\,\mathrm{d}W_t\,.\end{aligned}$$

We know from Lemma 2 that G, $\partial_1 h(F,\kappa)$, and F^2 are in $\mathcal{A}_2$, so

$$\mathbb{E}\left[\int_0^T \left(G_t + \frac{\sigma^2}{2}\,\partial_1 h(F_t,\kappa)\right)\,F_t^2\,\mathrm{d}t\right] = \left\langle G, F^2\right\rangle + \frac{\sigma^2}{2}\,\left\langle \partial_1 h(F,\kappa), F^2\right\rangle$$

is well-defined. Since

$$\mathbb{E}\left[\int_0^T (\partial_1 h(F_t,\kappa))^2\,F_t^4\,\mathrm{d}t\right] \leq \left\|(\partial_1 h(F,\kappa))^2\right\|\,\left\|F^4\right\| < \infty,$$

the process

$$\int_0^t \partial_1 h(F_s,\kappa)\, F_s^2\, \mathrm{d}W_s\,, \quad 0 \le t \le T\,,$$

is a martingale, so

$$\mathbb{E}\left[\int_0^T \partial_1 h(F_t,\kappa)\, F_t^2\, \mathrm{d}W_t\right] = 0\,.$$

Therefore, $\mathbb{E}[X_T]$ is well-defined, with

$$\mathbb{E}[X_T] = \varphi(h(F_0,\kappa),\kappa) - \mathbb{E}\left[\int_0^T \left(G_t + \frac{\sigma^2}{2}\,\partial_1 h(F_t,\kappa)\right)\, F_t^2\, \mathrm{d}t\right]\,.$$

On the other hand, (A15) implies

$$\Pi(F_t,\kappa) = \frac{\lambda\,\pi\,(v-\pi)\,F_t^2}{\partial_{11}\varphi\,(h(F_t,\kappa),\kappa)} = \lambda\,\pi\,(\pi - v)\,\partial_1 h(F_t,\kappa)\,F_t^2\,,$$

so

$$\mathbb{E}\left[\int_0^T \Pi(F_t,\kappa)\,\mathrm{d}t\right] = \lambda\,\pi\,(\pi-v)\,\mathbb{E}\left[\int_0^T \partial_1 h(F_t,\kappa)\,F_t^2\,\mathrm{d}t\right] = \lambda\,\pi\,(\pi-v)\,\left\langle \partial_1 h(F,\kappa), F^2\right\rangle$$

is well-defined. It follows that $\tilde{H}$ is well-defined and (26) can be written as

$$J[\nu^\star] + \mathbb{E}\left[\int_0^T \left\{\left(\frac{\sigma^2}{2} + \lambda\,\pi\,(\pi-v)\right)\,\partial_1 h(F_t,\kappa)\,F_t^2 + A_t\,F_t\,(Y_t - Y_0) - \tfrac{\phi}{2}\,(Y_t - Y_0)^2\right\}\,\mathrm{d}t\right]\,.$$

□

Appendix J. Proof of Proposition 7

By Proposition 6, it is enough to show $J[\nu^\star]$ and

$$\hat{H} \coloneqq \mathbb{E}\left[\int_0^T \left\{\left(\frac{\sigma^2}{2} + \lambda\,\pi\,(\pi-v)\right)\,\partial_1 h(F_t,\kappa)\,F_t^2 + A_t\,F_t\,(Y_t - Y_0) - \frac{\phi}{2}\,(Y_t - Y_0)^2\right\}\,\mathrm{d}t\right]\,,$$

are both continuous in κ. To that end, fix $\kappa_n \to \kappa$. Because

$$|Y_t(\kappa_n) - Y_t(\kappa)| = |h(F_t,\kappa_n) - h(F_t,\kappa)| \le \left(F_t^{\mathfrak{p}} + F_t^{\mathfrak{q}}\right)\,|\mathfrak{C}(\kappa_n) - \mathfrak{C}(\kappa)|\,,$$

we have

$$\begin{aligned}\|Y(\kappa_n) - Y(\kappa)\| = \mathbb{E}\left[\int_0^T |Y_t(\kappa_n) - Y_t(\kappa)|^2\,\mathrm{d}t\right]^{1/2} &\le |\mathfrak{C}(\kappa_n) - \mathfrak{C}(\kappa)|\,\mathbb{E}\left[\int_0^T \left(F_t^{\mathfrak{p}} + F_t^{\mathfrak{q}}\right)^2\,\mathrm{d}t\right]^{1/2} \\ &\le |\mathfrak{C}(\kappa_n) - \mathfrak{C}(\kappa)|\,\left(\|F^{\mathfrak{p}}\| + \|F^{\mathfrak{q}}\|\right)\end{aligned}$$

and

$$\|Y_0(\kappa_n) - Y_0(\kappa)\| \le |\mathfrak{C}(\kappa_n) - \mathfrak{C}(\kappa)| \left(\left\|F_0^{\mathfrak{p}}\right\| + \left\|F_0^{\mathfrak{q}}\right\|\right)$$

so the map $\kappa \mapsto Y(\kappa) - Y_0(\kappa)$ from $(0,\infty)$ to $\mathcal{A}_2$ is continuous. It follows that $\kappa \mapsto \hat{H}(\kappa)$ is continuous as the composition of

$$\zeta \mapsto \left(\frac{\sigma^2}{2} + \lambda\,\pi\,(\pi - v)\right) \left\langle \partial_1 h(F,\kappa), F^2\right\rangle + \langle A\,F, \zeta\rangle - \frac{\phi}{2}\,\|\zeta\|^2$$

with $\kappa \mapsto Y(\kappa) - Y_0(\kappa)$.

Next, we consider $J[\nu^\star]$. By Proposition 2 and Proposition 3, $\nu^\star = \Lambda^{-1} b$, so

$$J[\nu^\star] = -\frac{1}{2}\,\left\langle \Lambda\Lambda^{-1}b, \Lambda^{-1}b\right\rangle + \left\langle b, \Lambda^{-1}b\right\rangle = \frac{1}{2}\,\left\langle b, \Lambda^{-1}b\right\rangle\,,$$

where

$$b = \mathfrak{I}^\top(F\,G) + (c - \beta\,\mathfrak{I}^\top - \phi\,\mathfrak{Q}^\top)(Y - Y_0) + \mathfrak{Q}^\top(A\,F)\,.$$

Since

$$\begin{aligned}|G_t(\kappa_n) - G_t(\kappa)| &\le |A_t|\,|\partial_1 h(F_t,\kappa_n) - \partial_1 h(F_t,\kappa)| + \frac{\sigma^2}{2} F_t\,|\partial_{11} h(F_t,\kappa_n) - \partial_{11} h(F_t,\kappa)| \\ &\le \left(|A_t| + \frac{\sigma^2}{2}\,F_t\right)\,\left(F_t^{\mathfrak{p}} + F_t^{\mathfrak{q}}\right)\,|\mathfrak{C}(\kappa_n) - \mathfrak{C}(\kappa)|\,,\end{aligned}$$

we have

$$\begin{aligned}&\|\mathfrak{I}^\top(F\,G(\kappa_n)) - \mathfrak{I}^\top(F\,G(\kappa))\| \\ &\le \left\|\mathfrak{I}^\top\right\|_{\text{op}}\,\mathbb{E}\left[\int_0^T F_t^2\,|G_t(\kappa_n) - G_t(\kappa)|^2\,\mathrm{d}t\right]^{\frac{1}{2}} \\ &\le \left\|\mathfrak{I}^\top\right\|_{\text{op}}\,\mathbb{E}\left[\int_0^T \left(|A_t| + \frac{\sigma^2}{2}\,F_t\right)^2\,\left(F_t^{\mathfrak{p}+1} + F_t^{\mathfrak{q}+1}\right)^2\,|\mathfrak{C}(\kappa_n) - \mathfrak{C}(\kappa)|^2\,\mathrm{d}t\right]^{\frac{1}{2}} \\ &\le \left\|\mathfrak{I}^\top\right\|_{\text{op}}\left\{\mathbb{E}\,\left[\int_0^T |A_t|^p\,\mathrm{d}t\right]^{\frac{1}{p}}\,\left(\left\|F^{\frac{(\mathfrak{p}+1)\,p}{p-2}}\right\|^{\frac{p-2}{p}} + \left\|F^{\frac{(\mathfrak{q}+1)\,p}{p-2}}\right\|^{\frac{p-2}{p}}\right) + \frac{\sigma^2}{2}\,\left\|F^{\mathfrak{p}+2} + F^{\mathfrak{q}+2}\right\|\right\}\,|\mathfrak{C}(\kappa_n) - \mathfrak{C}(\kappa)|\,,\end{aligned}$$

so $\kappa \mapsto \mathfrak{I}^\top(F\,G(\kappa))$ is continuous and thus

$$\kappa \mapsto b(\kappa) = \mathfrak{I}^\top(F\,G(\kappa)) + (c - \beta\,\mathfrak{I}^\top - \phi\,\mathfrak{Q}^\top)(Y(\kappa) - Y_0(\kappa)) + \mathfrak{Q}^\top(A\,F)$$

is continuous. It follows that $\kappa \mapsto J[\nu^\star](\kappa) = \left\langle \Lambda^{-1}b(\kappa), b(\kappa)\right\rangle/2$ is continuous. □

Appendix K. Proof of Proposition 8

Recall that the stage-three trading volumes generate fee revenue (11). In the case of a CPM, these write

$$\Pi(F_t, \kappa) = \gamma\, \kappa\, \sqrt{F_t}\,,$$

where we define γ as in (30). In the no-replication case $\nu \equiv 0$, the value function is

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \Pi(F_t,\kappa)\,\mathrm{d}t + X_T + (Y_T - Y_0)\,F_T - \tfrac{\phi}{2}\int_0^T (Y_t - Y_0)^2\,\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T \gamma\,\kappa\,F_t^{1/2}\,\mathrm{d}t + 2\,\kappa\,F_T^{1/2} - \kappa\,F_0^{-1/2}\,F_T - \tfrac{\phi}{2}\int_0^T \kappa^2\,\left(F_t^{-1/2} - F_0^{-1/2}\right)^2\,\mathrm{d}t\right] \\
&= -\tfrac{\phi}{2}\,\mathbb{E}\left[\int_0^T \left(F_t^{-1/2} - F_0^{-1/2}\right)^2\,\mathrm{d}t\right]\kappa^2 + \mathbb{E}\left[\int_0^T \gamma\,F_t^{1/2}\,\mathrm{d}t + 2\,F_T^{1/2} - F_0^{-1/2}\,F_T\right]\kappa\,.
\end{aligned}$$

In this case the optimal supply of liquidity is

$$\underline{\kappa} = \frac{\mathbb{E}\left[\int_0^T \gamma\,F_t^{1/2}\,\mathrm{d}t + 2\,F_T^{1/2} - F_0^{-1/2}\,F_T\right]}{\phi\,\mathbb{E}\left[\int_0^T \left(F_t^{-1/2} - F_0^{-1/2}\right)^2\,\mathrm{d}t\right]}\,.$$

In the no-transient-impact case, the solutions in (24)–(A14)–(23) become

$$\begin{aligned}
\ell_t^\star &= -\kappa\,\underbrace{\tfrac{\phi}{2\,\eta}\mathbb{E}\left[\int_t^T \tilde{P}(t,s)\,F_s^{-1/2}\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right]}_{=:-C_t^\ell} + \underbrace{\tfrac{1}{2\,\eta}\mathbb{E}\left[\int_t^T \tilde{P}(t,s)\,A_s\,F_s\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right]}_{=:D_t^\ell} \\
Q_t^\star &= \kappa\,\underbrace{\int_0^t \tilde{P}(s,t)\,C_s^\ell\,\mathrm{d}s - F_0^{-1/2}\,\tilde{P}(0,t)}_{=:C_t^Q} + \underbrace{\int_0^t \tilde{P}(s,t)\,D_s^\ell\,\mathrm{d}s}_{=:D_t^Q} \\
\nu_t^\star &= \underbrace{\left(P(t)\,C_t^Q + C_t^\ell\right)}_{=:C_t^\nu}\,\kappa + \underbrace{P(t)\,D_t^Q + D_t^\ell}_{=:D_t^\nu}
\end{aligned}$$

When $\nu = \nu^\star$, the value function is

$$\begin{aligned}
&-\tfrac{\phi}{2}\,\mathbb{E}\left[\int_0^T \left(F_t^{-1/2} - F_0^{-1/2}\right)^2\,\mathrm{d}t\right]\kappa^2 + \mathbb{E}\left[\int_0^T \gamma\,F_t^{1/2}\,\mathrm{d}t + 2\,F_T^{1/2} - F_0^{-1/2}\,F_T\right]\kappa \\
&\quad + \mathbb{E}\left[(Q_T^\star + Y_0)\;F_T - \int_0^T (F_t + \eta\,\nu_t^\star)\;\nu_t^\star\,\mathrm{d}t - \tfrac{\phi}{2}\int_0^T \left((Q_t^\star + Y_0)^2 + 2\;(Q_t^\star + Y_0)\;(Y_t - Y_0)\right)\mathrm{d}t\right] \\
&= -\tfrac{\phi}{2}\,\mathbb{E}\left[\int_0^T \left(F_t^{-1/2} - F_0^{-1/2}\right)^2\,\mathrm{d}t\right]\kappa^2 + \mathbb{E}\left[\int_0^T \gamma\,F_t^{1/2}\,\mathrm{d}t + 2\,F_T^{1/2} - F_0^{-1/2}\,F_T\right]\kappa \\
&\quad + \mathbb{E}\Bigg[\left(C_T^Q\,\kappa + D_T^Q + F_0^{-1/2}\,\kappa\right)\;F_T - \int_0^T (F_t + \eta\;(C_t^\nu\,\kappa + D_t^\nu))\,(C_t^\nu\,\kappa + D_t^\nu)\;\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\qquad -\tfrac{\phi}{2}\int_0^T \left(\left(C_t^Q\,\kappa + D_t^Q + F_0^{-1/2}\,\kappa\right)^2 + 2\left(C_t^Q\,\kappa + D_t^Q + F_0^{-1/2}\,\kappa\right)\left(F_t^{-1/2} - F_0^{-1/2}\right)\kappa\right)\mathrm{d}t\Bigg] \\
&= -\tfrac{\phi}{2}\,\mathbb{E}\left[\int_0^T \left(F_t^{-1/2} - F_0^{-1/2}\right)^2 \mathrm{d}t\right]\kappa^2 + \mathbb{E}\left[\int_0^T \gamma\,F_t^{1/2}\,\mathrm{d}t + 2\,F_T^{1/2} - F_0^{-1/2}\,F_T\right]\kappa \\
&\quad - \mathbb{E}\left[\int_0^T \left(\eta\,(C_t^\nu)^2 + \tfrac{\phi}{2}\left(C_t^Q + F_0^{-1/2}\right)^2 + \phi\left(C_t^Q + F_0^{-1/2}\right)\left(F_t^{-1/2} - F_0^{-1/2}\right)\right)\mathrm{d}t\right]\kappa^2 \\
&\quad + \mathbb{E}\left[\left(C_T^Q + F_0^{-1/2}\right)F_T - \int_0^T \left(C_t^\nu\,F_t + 2\,\eta\,C_t^\nu\,D_t^\nu + \phi\,D_t^Q\left(C_t^Q + F_t^{-1/2}\right)\right)\mathrm{d}t\right]\kappa \\
&\quad + \mathbb{E}\left[D_T^Q\,F_T - \int_0^T \left(F_t\,D_t^\nu + \eta\,(D_t^\nu)^2 + \tfrac{\phi}{2}\left(D_t^Q\right)^2\right)\mathrm{d}t\right].
\end{aligned}
$$

In this case the optimal κ is

$$
\begin{aligned}
\kappa^\star &= \frac{\mathfrak{A} + \mathbb{E}\left[\int_0^T \gamma\,F_t^{1/2}\,\mathrm{d}t + 2\,F_T^{1/2} - F_0^{-1/2}\,F_T\right]}{\phi\left(\mathfrak{B} + \mathbb{E}\left[\int_0^T \left(F_t^{-1/2} - F_0^{-1/2}\right)^2 \mathrm{d}t\right]\right)} \\
&= \left(\underline{\kappa} + \frac{\mathfrak{A}}{\phi\,\mathbb{E}\left[\int_0^T \left(F_t^{-1/2} - F_0^{-1/2}\right)^2 \mathrm{d}t\right]}\right)\frac{\mathbb{E}\left[\int_0^T \left(F_t^{-1/2} - F_0^{-1/2}\right)^2 \mathrm{d}t\right]}{\mathfrak{B} + \mathbb{E}\left[\int_0^T \left(F_t^{-1/2} - F_0^{-1/2}\right)^2 \mathrm{d}t\right]}
\end{aligned}
$$

where

$$
\mathfrak{A} \coloneqq \mathbb{E}\left[\left(C_T^Q + F_0^{-1/2}\right)F_T - \int_0^T \left(C_t^\nu\,F_t + 2\,\eta\,C_t^\nu\,D_t^\nu + \phi\,D_t^Q\left(C_t^Q + F_t^{-1/2}\right)\right)\mathrm{d}t\right],
$$

and

$$
\mathfrak{B} \coloneqq \mathbb{E}\left[\int_0^T \left(\tfrac{2\,\eta}{\phi}\,(C_t^\nu)^2 + \left(C_t^Q + F_0^{-1/2}\right)^2 + 2\left(C_t^Q + F_0^{-1/2}\right)\left(F_t^{-1/2} - F_0^{-1/2}\right)\right)\mathrm{d}t\right].
$$

We have

$$
\begin{aligned}
C_t^\ell &= -\tfrac{\phi}{2\,\eta}\mathbb{E}\left[\int_t^T \tilde{P}(t,s)\,F_s^{-1/2}\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&= -\tfrac{\phi}{2\,\eta}\,\tilde{P}(t,0)\,\mathbb{E}\left[\int_t^T \tilde{P}(0,s)\,F_s^{-1/2}\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&= -\tfrac{\phi}{2\,\eta}\,\tilde{P}(t,0)\left(\tilde{M}_t - \int_0^t \tilde{P}(0,s)\,F_s^{-1/2}\mathrm{d}s\right),
\end{aligned}
$$

Then generalized Itô's formula gives

$$
\begin{aligned}
\mathrm{d}C_t^\ell &= -P(t)\,C_t^\ell\,\mathrm{d}t - \tfrac{\phi}{2\,\eta}\,\tilde{P}(t,0)\left(\mathrm{d}\tilde{M}_t - \tilde{P}(0,t)\,F_t^{-1/2}\,\mathrm{d}t\right) \\
&= \left(-P(t)\,C_t^\ell + \tfrac{\phi}{2\,\eta}\,F_t^{-1/2}\right)\mathrm{d}t - \tfrac{\phi}{2\,\eta}\,\tilde{P}(t,0)\,\mathrm{d}\tilde{M}_t.
\end{aligned}
$$

Since

$$
\begin{aligned}
\mathrm{d}C_t^\nu &= P'(t)\,C_t^Q\,\mathrm{d}t + P(t)\,C_t^\nu\,\mathrm{d}t + \mathrm{d}C_t^\ell \\
&= \left(P'(t)\,C_t^Q + P(t)\,\left(P(t)\,C_t^Q + C_t^\ell\right) - P(t)\,C_t^\ell + \tfrac{\phi}{2\,\eta}\,F_t^{-1/2}\right)\,\mathrm{d}t - \tfrac{\phi}{2\,\eta}\,\tilde{P}(t,0)\,\mathrm{d}\tilde{M}_t \\
&= \left(\left(P'(t) + P(t)^2\right)\,C_t^Q + \tfrac{\phi}{2\,\eta}\,F_t^{-1/2}\right)\,\mathrm{d}t - \tfrac{\phi}{2\,\eta}\,\tilde{P}(t,0)\,\mathrm{d}\tilde{M}_t \\
&= \tfrac{\phi}{2\,\eta}\,\left(C_t^Q + F_t^{-1/2}\right)\,\mathrm{d}t - \tfrac{\phi}{2\,\eta}\,\tilde{P}(t,0)\,\mathrm{d}\tilde{M}_t\,,
\end{aligned} \tag{A16}
$$

where the last equality uses (A12), and

$$
\mathrm{d}\left(C_t^\nu\,D_t^Q\right) = \left[C_t^\nu\,D_t^\nu + \tfrac{\phi}{2\,\eta}\,D_t^Q\,\left(C_t^Q + F_t^{-1/2}\right)\right]\,\mathrm{d}t - \tfrac{\phi}{2\,\eta}\,\tilde{P}(t,0)\,D_t^Q\,\mathrm{d}\tilde{M}_t\,,
$$

we have

$$
\begin{aligned}
\mathfrak{A} &= \mathbb{E}\left[\left(C_T^Q + F_0^{-1/2}\right)\,F_T - \int_0^T \left(C_t^\nu\,F_t + 2\,\eta\,C_t^\nu\,D_t^\nu + \phi\,D_t^Q\,\left(C_t^Q + F_t^{-1/2}\right)\right)\,\mathrm{d}t\right] \\
&= \mathbb{E}\Bigg[\int_0^T \left(C_t^\nu\,F_t + \left(C_t^Q + F_0^{-1/2}\right)\,A_t\,F_t\right)\mathrm{d}t + \sigma\int_0^T \left(C_t^Q + F_0^{-1/2}\right)\,F_t\,\mathrm{d}W_t \\
&\qquad - \int_0^T \left(C_t^\nu\,F_t + 2\,\eta\,C_t^\nu\,D_t^\nu + \phi\,D_t^Q\,\left(C_t^Q + F_t^{-1/2}\right)\right)\,\mathrm{d}t\Bigg] \\
&= \mathbb{E}\left[\int_0^T \left(\left(C_t^Q + F_0^{-1/2}\right)\,A_t\,F_t - 2\,\eta\,C_t^\nu\,D_t^\nu - \phi\,D_t^Q\,\left(C_t^Q + F_t^{-1/2}\right)\right)\,\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T \left(C_t^Q + F_0^{-1/2}\right)\,A_t\,F_t\,\mathrm{d}t - 2\,\eta\,(C_T^\nu\,D_T^Q - C_0^\nu\,D_0^Q) - \phi\int_0^T \tilde{P}(t,0)\,D_t^Q\,\mathrm{d}\tilde{M}_t\right] \\
&= \mathbb{E}\left[\int_0^T \left(C_t^Q + F_0^{-1/2}\right)\,A_t\,F_t\,\mathrm{d}t\right]\,,
\end{aligned}
$$

where the term $\mathbb{E}\left[\int_0^T \left(C_t^Q + F_0^{-1/2}\right)\,F_t\,\mathrm{d}W_t\right]$ vanishes because

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \left|C_t^Q + F_0^{-1/2}\right|^2\,F_t^2\,\mathrm{d}t\right] \\
&\lesssim \mathbb{E}\left[\int_0^T \left|C_t^Q\right|^2\,F_t^2\,\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T F_t^2\,\mathrm{d}t\right] \\
&\leq \mathbb{E}\left[\int_0^T \left|C_t^Q\right|^4\,\mathrm{d}t\right]^{1/2}\,\mathbb{E}\left[\int_0^T F_t^4\,\mathrm{d}t\right]^{1/2} + \mathbb{E}\left[\int_0^T F_t^2\,\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T \left|\int_0^t \tilde{P}(s,t)\,C_s^\ell\,\mathrm{d}s - F_0^{-1/2}\,\tilde{P}(0,t)\right|^4\,\mathrm{d}t\right]^{1/2}\,\mathbb{E}\left[\int_0^T F_t^4\,\mathrm{d}t\right]^{1/2} + \mathbb{E}\left[\int_0^T F_t^2\,\mathrm{d}t\right] \\
&\lesssim \left(\mathbb{E}\left[\int_0^T \left|C_t^\ell\right|^4\,\mathrm{d}t\right] + F_0^{-2}\right)^{1/2}\,\mathbb{E}\left[\int_0^T F_t^4\,\mathrm{d}t\right]^{1/2} + \mathbb{E}\left[\int_0^T F_t^2\,\mathrm{d}t\right] \\
&\leq \left(\int_0^T \mathbb{E}\left[\left|\int_t^T \tilde{P}(t,s)\,F_s^{-1/2}\,\mathrm{d}s\right|^4\right]\,\mathrm{d}t + F_0^{-2}\right)^{1/2}\,\mathbb{E}\left[\int_0^T F_t^4\,\mathrm{d}t\right]^{1/2} + \mathbb{E}\left[\int_0^T F_t^2\,\mathrm{d}t\right]
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \left(\mathbb{E}\left[\int_0^T F_t^{-2}\,\mathrm{d}t\right] + F_0^{-2}\right)^{1/2} \mathbb{E}\left[\int_0^T F_t^{4}\,\mathrm{d}t\right]^{1/2} + \mathbb{E}\left[\int_0^T F_t^{2}\,\mathrm{d}t\right] \\
&< \infty\,.
\end{aligned}
$$

Next, we simplify $\mathfrak{B}$. By (A16),

$$
\mathrm{d}\left(C_t^\nu\left(C_t^Q + F_0^{-1/2}\right)\right) = \left[(C_t^\nu)^2 + \tfrac{\phi}{2\,\eta}\left(C_t^Q + F_0^{-1/2}\right)\left(C_t^Q + F_t^{-1/2}\right)\right]\mathrm{d}t - \tfrac{\phi}{2\,\eta}\,\tilde{P}(t,0)\left(C_t^Q + F_0^{-1/2}\right)\mathrm{d}\tilde{M}_t\,,
$$

It follows that

$$
\begin{aligned}
\mathfrak{B} &= \mathbb{E}\left[\int_0^T \left(\tfrac{2\,\eta}{\phi}\,(C_t^\nu)^2 + \left(C_t^Q + F_0^{-1/2}\right)^2 + 2\left(C_t^Q + F_0^{-1/2}\right)\left(F_t^{-1/2} - F_0^{-1/2}\right)\right)\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T \left(\tfrac{2\,\eta}{\phi}\,(C_t^\nu)^2 + \left(C_t^Q + F_0^{-1/2}\right)\left(C_t^Q + F_t^{-1/2}\right) + \left(C_t^Q + F_0^{-1/2}\right)\left(F_t^{-1/2} - F_0^{-1/2}\right)\right)\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T \left(C_t^Q + F_0^{-1/2}\right)\left(F_t^{-1/2} - F_0^{-1/2}\right)\mathrm{d}t\right]
\end{aligned}
$$

On the other hand,

$$
\mathfrak{B} + \mathbb{E}\left[\int_0^T \left(F_t^{-1/2} - F_0^{-1/2}\right)^2\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T \left(\tfrac{2\,\eta}{\phi}\,(C_t^\nu)^2 + \left(C_t^Q + F_t^{-1/2}\right)^2\right)\mathrm{d}t\right] \geq 0\,.
$$

□

Appendix L. Proof of Lemma 4

It is enough to show $\mathbb{E}\left[\int_0^t \tilde{P}(t,0)^2\,(D_t^Q)^2\,\mathrm{d}\langle\tilde{M}\rangle_t\right] < \infty$ and $\mathbb{E}\left[\int_0^t \tilde{P}(t,0)^2\,(C_t^Q)^2\,\mathrm{d}\langle\tilde{M}\rangle_t\right] < \infty$. If $t \leq s$, we have

$$
F_s = F_0\,e^{\int_0^s\left(A_u - \frac{\sigma^2}{2}\right)\mathrm{d}u + \sigma\,W_s} = F_t\,e^{-\frac{\sigma^2}{2}\,(s-t)}\,e^{\int_t^s A_u\,\mathrm{d}u + \sigma\,(W_s - W_t)}\,.
$$

A has the representation

$$
A_u = \mu + (A_t - \mu)\,e^{-\theta\,(u-t)} + \xi\int_t^u e^{-\theta\,(u-r)}\mathrm{d}W_r \qquad t \leq u\,.
$$

Since the integrand is jointly continuous, deterministic, and bounded, the stochastic Fubini theorem implies

$$
\begin{aligned}
\int_t^s A_u\,\mathrm{d}u &= \mu\,(s-t) + (A_t - \mu)\,\frac{1 - e^{-\theta\,(s-t)}}{\theta} + \xi\int_t^s\int_t^u e^{-\theta\,(u-r)}\,\mathrm{d}W_r\,\mathrm{d}u \\
&= \mu\,(s-t) + (A_t - \mu)\,\frac{1 - e^{-\theta\,(s-t)}}{\theta} + \xi\int_t^s\int_r^s e^{-\theta\,(u-r)}\,\mathrm{d}u\,\mathrm{d}W_r \\
&= \mu\,(s-t) + (A_t - \mu)\,\frac{1 - e^{-\theta\,(s-t)}}{\theta} + \frac{\xi}{\theta}\int_t^s \left(1 - e^{-\theta\,(s-r)}\right)\mathrm{d}W_r\,.
\end{aligned}
$$

Then

$$\mathbb{E}\left[F_s^{-1/2}\,\middle|\,\mathcal{F}_t\right] = F_t^{-1/2}\,e^{\left(-\frac{\mu}{2}+\frac{\sigma^2}{4}\right)(s-t)-\frac{(A_t-\mu)\left(1-e^{-\theta\,(s-t)}\right)}{2\,\theta}}\,\mathbb{E}\left[e^{-\frac{1}{2}\int_t^s\left(\frac{\xi}{\theta}+\sigma-\frac{\xi}{\theta}\,e^{-\theta\,(s-r)}\right)\mathrm{d}W_r}\right],$$

where the quantity $-\frac{1}{2}\int_t^s\left(\frac{\xi}{\theta}+\sigma-\frac{\xi}{\theta}\,e^{-\theta\,(s-r)}\right)\mathrm{d}W_r$, viewed as a Wiener integral, is a Gaussian random variable with mean zero and variance

$$\tfrac{1}{4}\int_t^s\left(\tfrac{\xi}{\theta}+\sigma-\tfrac{\xi}{\theta}\,e^{-\theta\,(s-r)}\right)^2\mathrm{d}r\,,$$

so

$$\begin{aligned}\mathbb{E}\left[F_s^{-1/2}\,\middle|\,\mathcal{F}_t\right] &= F_t^{-1/2}\,e^{-\frac{A_t\left(1-e^{-\theta\,(s-t)}\right)}{2\,\theta}}\,e^{\left(-\frac{\mu}{2}+\frac{\sigma^2}{4}\right)(s-t)+\frac{\mu\left(1-e^{-\theta\,(s-t)}\right)}{2\,\theta}+\frac{1}{8}\int_t^s\left(\frac{\xi}{\theta}+\sigma-\frac{\xi}{\theta}\,e^{-\theta\,(s-r)}\right)^2\mathrm{d}r}\\ &= F_t^{-1/2}\,e^{-A_t\,g(s,t)}\,h(s,t)\,,\end{aligned}$$

where

$$g(s,t) \coloneqq \frac{\left(1-e^{-\theta\,(s-t)}\right)}{2\,\theta}$$

and

$$h(s,t) \coloneqq e^{\left(-\frac{\mu}{2}+\frac{\sigma^2}{4}\right)(s-t)+\frac{\mu\left(1-e^{-\theta\,(s-t)}\right)}{2\,\theta}+\frac{1}{8}\int_t^s\left(\frac{\xi}{\theta}+\sigma-\frac{\xi}{\theta}\,e^{-\theta\,(s-r)}\right)^2\mathrm{d}r}\,.$$

Thus

$$\begin{aligned}\tilde{M}_t &= \int_0^t \tilde{P}(0,s)\,F_s^{-1/2}\,\mathrm{d}s + F_t^{-1/2}\int_t^T \tilde{P}(0,s)\,e^{-A_t\,g(s,t)}\,h(s,t)\,\mathrm{d}s\\ &= \int_0^t \tilde{P}(0,s)\,F_s^{-1/2}\,\mathrm{d}s + F_t^{-1/2}\,H(A_t,t)\end{aligned} \tag{A17}$$

where

$$H(a,t) \coloneqq \int_t^T \tilde{P}(0,s)\,e^{-a\,g(s,t)}\,h(s,t)\,\mathrm{d}s\,.$$

Note that H is smooth with

$$\partial_1 H(a,t) = -\int_t^T \tilde{P}(0,s)\,e^{-a\,g(s,t)}\,g(s,t)\,h(s,t)\,\mathrm{d}s$$

Applying Itô to (A17) and using the fact that all finite variation terms must vanish since $\tilde{M}$ is a martingale give

$$\mathrm{d}\tilde{M}_t = F_t^{-1/2}\,\left(\xi\,\partial_1 H(A_t,t) - \frac{\sigma}{2}\,H(A_t,t)\right)\mathrm{d}W_t\,.$$

For any $q \geq 1$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \left|\xi\,\partial_1 H(A_t,t) - \frac{\sigma}{2}\,H(A_t,t)\right|^q \,\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T \left|\int_t^T \tilde{P}(0,s)\,e^{-A_t\,g(s,t)}\,\left(\xi\,g(s,t) + \frac{\sigma}{2}\right)\,h(s,t)\,\mathrm{d}s\right|^q \,\mathrm{d}t\right] \\
&\lesssim \mathbb{E}\left[\int_0^T \int_0^T e^{-q\,g(s,t)\,A_t}\,\mathrm{d}s\,\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T \int_0^T e^{-q\,g(s,t)\,\left(\mu+(A_0-\mu)\,e^{-\theta\,t}+\xi\int_0^t e^{-\theta\,(t-r)}\mathrm{d}W_r\right)}\,\mathrm{d}s\,\mathrm{d}t\right] \\
&= \int_0^T \int_0^T e^{-q\,g(s,t)\,\left(\mu+(A_0-\mu)\,e^{-\theta\,t}\right)}\,\mathbb{E}\left[e^{-q\,g(s,t)\,\xi\int_0^t e^{-\theta\,(t-r)}\mathrm{d}W_r}\right]\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^T \int_0^T e^{-q\,g(s,t)\,\left(\mu+(A_0-\mu)\,e^{-\theta\,t}\right)+\frac{1}{2}\,q^2\,g(s,t)^2\,\xi^2\,\int_0^t e^{-2\,\theta\,(t-r)}\,\mathrm{d}r}\,\mathrm{d}s\,\mathrm{d}t \\
&< \infty\,.
\end{aligned}
$$

Now take $q \in (2,p)$ and $r, s > 1$ such that $\frac{2}{q} + \frac{1}{r} + \frac{1}{s} = 1$, then

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t \tilde{P}(t,0)^2\,(D_t^Q)^2\,\mathrm{d}\langle\tilde{M}\rangle_t\right] \\
&= \mathbb{E}\left[\int_0^t \tilde{P}(t,0)^2\,(D_t^Q)^2\,F_t^{-1}\,\left(\xi\,\partial_1 H(A_t,t) - \frac{\sigma}{2}\,H(A_t,t)\right)^2\,\mathrm{d}t\right] \\
&\lesssim \mathbb{E}\left[\int_0^t (D_t^Q)^2\,F_t^{-1}\,\left(\xi\,\partial_1 H(A_t,t) - \frac{\sigma}{2}\,H(A_t,t)\right)^2\,\mathrm{d}t\right] \\
&\leq \mathbb{E}\left[\int_0^t |C_t^Q|^q\,\mathrm{d}t\right]^{\frac{2}{q}}\,\mathbb{E}\left[\int_0^t F_t^{-r}\,\mathrm{d}t\right]^{\frac{1}{r}}\,\mathbb{E}\left[\int_0^T \left|\xi\,\partial_1 H(A_t,t) - \frac{\sigma}{2}\,H(A_t,t)\right|^{2\,s}\,\mathrm{d}t\right]^{\frac{1}{s}} \\
&\lesssim \mathbb{E}\left[\int_0^t |A_t\,F_t|^q\,\mathrm{d}t\right]^{\frac{2}{q}}\,\mathbb{E}\left[\int_0^t F_t^{-r}\,\mathrm{d}t\right]^{\frac{1}{r}}\,\mathbb{E}\left[\int_0^T \left|\xi\,\partial_1 H(A_t,t) - \frac{\sigma}{2}\,H(A_t,t)\right|^{2\,s}\,\mathrm{d}t\right]^{\frac{1}{s}} \\
&\lesssim \mathbb{E}\left[\int_0^t |A_t|^p\,\mathrm{d}t\right]^{\frac{2}{p}}\,\mathbb{E}\left[\int_0^t F_t^{\frac{p\,q}{p-q}}\,\mathrm{d}t\right]^{\frac{2\,(p-q)}{p\,q}}\,\mathbb{E}\left[\int_0^t F_t^{-r}\,\mathrm{d}t\right]^{\frac{1}{r}}\,\mathbb{E}\left[\int_0^T \left|\xi\,\partial_1 H(A_t,t) - \frac{\sigma}{2}\,H(A_t,t)\right|^{2\,s}\,\mathrm{d}t\right]^{\frac{1}{s}} \\
&< \infty\,,
\end{aligned}
$$

and similarly,

$$
\mathbb{E}\left[\int_0^t \tilde{P}(t,0)^2\,(C_t^Q)^2\,\mathrm{d}\langle\tilde{M}\rangle_t\right] < \infty\,.
$$

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