Market Microstructure and Algorithmic Trading

University of Oxford

Fayçal Drissi

faycal.drissi@eng.ox.ac.uk
http://www.faycaldrissi.com

March 2, 2026

- An agent holds an **initial position** Q_0 at time $t = 0$ that they wish to **unwind** over a time window $[0, T]$.

- The trader's **inventory** over $[0, T]$ is modelled by the process $(Q_t)_{t \in [0,T]}$ with dynamics

$$\boxed{dQ_t = \nu_t \, dt}$$

- At time t, ν_t is the **trading velocity**, or **trading speed**, or **instantaneous trading volume**.

- **Admissible** strategies: $(\nu_t)_{t \in [0,T]}$ progressively measurable and satisfying the unwind constraint

$$\int_0^T \nu_t \, dt = -Q_0.$$

- Trading activity has a lasting effect on the price.

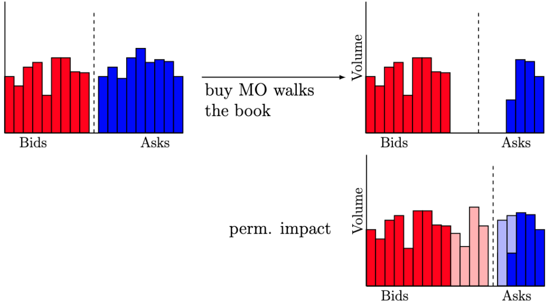


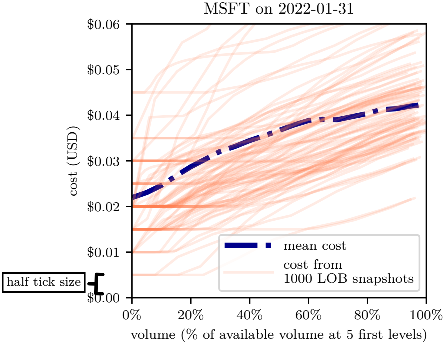
Figure 7: In the first two panels, an MO walks the book so the next midprice exhibits the temporary price impact. Immediately after the MO, market participants replenish the LOB. The difference between the midprice in the last panel and that of the first panel is the permanent impact.

- The **mid-price** of the asset has a **linear permanent impact** component and is modelled by the (controlled) process $(S_t)_{t \in [0,T]}$ with dynamics

$$\boxed{dS_t = k\,\nu_t\,dt + \sigma\,dW_t}$$

- The **market volume** V is flat and constant.
- **Execution costs**: the price obtained for each asset at time t is

$$\tilde{S}_t = S_t + \eta\, \nu_t / V.$$



- The cash of the agent:

$$dX_t = -\nu_t\, \tilde{S}_t\, dt = -\nu_t \left(S_t + \eta\, \nu_t / V \right) dt.$$

The **optimisation** problem

Find an optimal **deterministic** liquidation strategy ν that maximises the mean-variance criterion

$$\mathbb{E}[X_T] - \frac{\gamma}{2}\mathbb{V}[X_T],$$

where $\gamma > 0$ is the risk aversion parameter.

We compute the terminal wealth. For a strategy ν,

$$\begin{aligned} X_T =& X_0 - \int_0^T \nu_t\, S_t\, dt - \eta \int_0^T \frac{\nu_t^2}{V}\, dt \\ =& X_0 + Q_0\, S_0 + \int_0^T k\, \nu_t\, Q_t\, dt + \sigma \int_0^T Q_t\, dW_t - \eta \int_0^T \frac{\nu_t^2}{V}\, dt, \\ =& X_0 + Q_0\, S_0 - \frac{k}{2} Q_0^2 + \sigma \int_0^T Q_t\, dW_t - \eta \int_0^T \frac{\nu_t^2}{V}\, dt. \end{aligned}$$

The terminal wealth X_T is normally distributed with mean and variance

$$\begin{cases} \mathbb{E}[X_T] & = \underbrace{X_0 + Q_0 S_0}_{\text{MtM}} - \underbrace{\frac{k}{2} Q_0^2}_{\text{perm. impact.}} - \underbrace{\eta \int_0^T \frac{\nu_t^2}{V} dt}_{\text{execution costs}} \\ \mathbb{V}[X_T] & = \sigma^2 \int_0^T Q_t^2 \, dt. \end{cases}$$

The variance is **increasing** in σ and in Q. It is minimised when the agent trades quickly.
The expected wealth is maximized by trading slowly.

The objective of the agent is to minimise

$$\eta \int_0^T \frac{\nu_t^2}{V} dt + \frac{\gamma}{2} \sigma^2 \int_0^T Q_t^2 \, dt.$$

Permanent impact costs cannot be avoided !

The problem is equivalent to minimising the functional J over deterministic absolutely continuous functions:

$$\boxed{J(Q) = \int_0^T \left(\eta \, \frac{Q'(t)^2}{V} + \frac{\gamma}{2} \sigma^2 \, Q_t^2 \right) dt \,,}$$

with constraints $Q(0) = Q_0$ and $Q(T) = 0$.

Reminder: If $x^\star$ is a minimizer of the functional

$$J : x \mapsto \int_0^T \left(f(x(t)) + g(x'(t))\right) dt,$$

Then it is characterized by the Hamiltonian system

$$\begin{cases} p^{'}(t) & = f^{'}(x^*(t)), \\ x^{*'}(t) & = \tilde{g}^{'}(p(t)), \\ x^*(0) & = a, \\ x^*(T) & = b. \end{cases}$$

where $\tilde{g}$ is the Legendre-Fenchel transform of g, i.e.,

$$\tilde{g} : p \in \mathbb{R}^d \mapsto \sup_{x \in \mathbb{R}^d} p \cdot x - g(x).$$

Our problem:

$$\boxed{J\left(Q\right) = \int_0^T \left(\eta \, \frac{Q^{'}\left(t\right)^2}{V} + \frac{\gamma}{2}\sigma^2 Q_t^2\right) \, dt \, ,}$$

The Legendre-Fenchel transform of $g : x \mapsto \eta\, x^2/V$ is $\tilde{g} : p \mapsto \frac{p^2\, V}{4\,\eta}$

The Hamiltonian system characterising the optimal trading curve Q^* is and

$$\begin{cases} p'(t) &= \gamma\,\sigma^2\, Q^\star(t)\,, \\ Q^{\star'}(t) &= V\,p(t)/2\,\eta, \\ Q^\star(0) &= Q_0, \\ Q^\star(T) &= 0. \end{cases} \implies Q^{\star''}(t) = V\,\frac{\gamma\,\sigma^2}{2\,\eta}\,Q^\star(t)\,,$$

The **solution** is

$$\boxed{Q^\star(t) = Q_0\,\frac{\sinh\left((T-t)\sqrt{\frac{\gamma\,V\,\sigma^2}{2\,\eta}}\right)}{\sinh\left(T\sqrt{V\,\frac{\gamma\,\sigma^2}{2\,\eta}}\right)} \implies \nu^\star(t) = -Q_0\sqrt{\frac{\gamma\,V\,\sigma^2}{2\,\eta}}\,\frac{\cosh\left((T-t)\sqrt{\frac{\gamma\,V\,\sigma^2}{2\,\eta}}\right)}{\sinh\left(T\sqrt{V\,\frac{\gamma\,\sigma^2}{2\,\eta}}\right)}}$$

Recall the discrete-time counterpart:

$$\begin{cases} Q_n^\star &= \frac{Q_0\ \sinh(\alpha(T-t_n))}{\sinh(\alpha T)} \\ 2\,\cosh(\alpha\,\Delta t) &= \frac{\gamma\,\sigma^2\,V}{2\,\tilde{\eta}}\Delta t^2 \end{cases} \xrightarrow[\Delta t \to 0]{\text{Taylor expansion}} \begin{cases} Q_t^\star &= \frac{Q_0\ \sinh(\alpha(T-t))}{\sinh(\alpha T)} \\ \alpha &= \sqrt{\frac{\gamma\,\sigma^2\,V}{2\,\tilde{\eta}}}. \end{cases}$$

Deterministic strategies are computed at the start of the liquidation, and stay the same for whichever price path.

In practice, execution algorithms are in **two layers**: the first is strategic and defines the optimal trading curve. The second is **tactical** and tracks the optimal trading curve with different type of orders, different trading venues, etc.

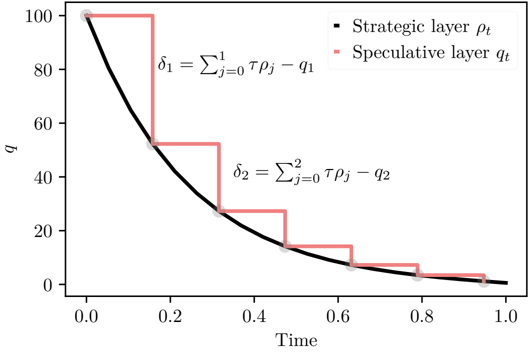


Figure: Execution algorithms in two layers.

Effect of parameters

- **liquidity parameters** η and V: they are **scaling** factors; the larger η, the more the agent pays costs, the lower V, the more the agent pays costs, so the liquidation process is slower:

$$\frac{dQ^\star}{d\frac{\eta}{V}}/Q_0 \geq 0.$$

- **volatility** σ: importance of price risk in the performance; the larger σ, the faster the liquidation to reduce exposure to risk:

$$\frac{dQ^\star}{d\sigma}/Q_0 \leq 0.$$

- **risk aversion** γ: balances the tradeoff between costs and price risk; the larger γ, the more sensitive the agent to price risk. Thus, high γ means fast execution:

$$\frac{dQ^\star}{d\gamma}/Q_0 \leq 0.$$

  Note that no risk aversion leads to TWAP:

$$\lim_{\gamma\to 0} Q^\star(t) = Q_0\left(1 - \frac{t}{T}\right)$$

Effect of parameters

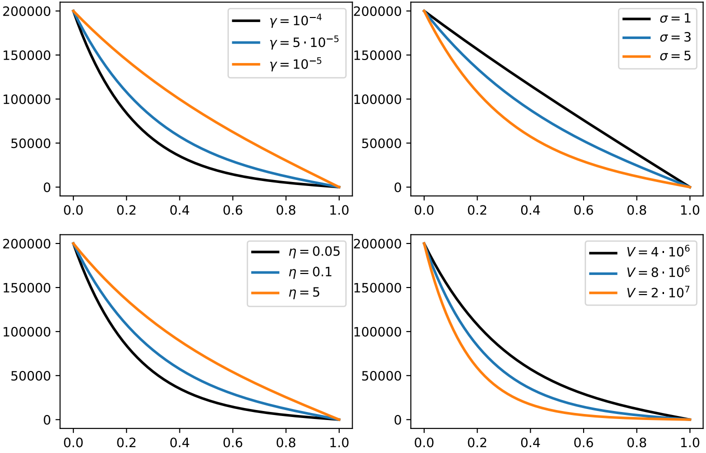


Figure: Optimal trading curves.

Discussion based on the work of (Gatheral 2010).

- Assume the permanent impact is $\kappa(\nu)$ and for simplicity, assume zero execution costs. The dynamics are

$$\begin{cases} dQ_t &= \nu_t \, dt \\ dS_t &= \sigma \, dW_t + \kappa(\nu_t) \, dt \\ dX_t &= -\nu_t \, S_t \, dt. \end{cases}$$

- **Definition**: there is dynamic arbitrage if $\exists t_1 < t_2$ and a **roundtrip** strategy ν such that

$$\begin{cases} \int_{t_1}^{t_2} \nu_t \, dt &= 0 \\ \mathbb{E}\left[X_{t_2} \middle| \mathcal{F}_{t_1}\right] &> X_{t_1}. \end{cases}$$

Theorem: $\kappa(\cdot)$ linear is the only possible choice to guarantee absence of dynamic arbitrage.

proof:

- consider the roundtrip strategy $\nu_t = \begin{cases} a & \text{if } t \in \left[t_1, \frac{a\,t_1 + b\,t_2}{a+b}\right] \\ -b & \text{if } t \in \left[\frac{a\,t_1 + b\,t_2}{a+b}, t_2\right] \end{cases}$.

  Then we can show that

$$\mathbb{E}\left[X_{t_2} \middle| \mathcal{F}_{t_1}\right] = X_{t_1} + \frac{a\,b}{2\,(a+b)^2}\,(t_2 - t_1)^2\,(b\,\kappa\,(a) + a\,\kappa\,(-b))\,.$$

$$\begin{aligned}
\text{No dyn. arb.} \implies & \mathbb{E}\left[X_{t_2} \middle| \mathcal{F}_{t_1}\right] \le X_{t_1}\,, \quad \forall a, b \\
\implies & \begin{cases} b\,\kappa\,(a) + a\,\kappa\,(-b) \le 0 \\ a\,\kappa\,(-b) + b\,\kappa\,(a) \ge 0 \quad \text{(replace } (a,b) \text{ by } (-b,-a)\text{)} \end{cases} \\
\implies & b\,\kappa\,(a) = -a\,\kappa\,(-b)\,, \quad \forall a, b \\
\implies & \kappa\,(a) = -\kappa\,(-a)\,,\ \forall a \quad \text{for } b = a \\
\implies & \kappa\,(a) = -a\,\text{sign}\,(a)\ \kappa\,(-\text{sign}\,(a)) = a\,\kappa\,(1) \quad \text{for } b = \text{sign}\,(a) \neq 0
\end{aligned}$$

proof:

- We need to show $\kappa(0) = 0$.

Choose

$$\nu_t = \begin{cases} \kappa(0) & \text{if } t \in \left[t_1, t_1 + \frac{t_2 - t_1}{3}\right] \\ 0 & \text{if } t \in \left[t_1 + \frac{t_2 - t_1}{3}, t_1 + 2\frac{t_2 - t_1}{3}\right] \\ -\kappa(0) & \text{if } t \in \left[t_1 + 2\frac{t_2 - t_1}{3}, t_2\right] \end{cases}$$

and obtain

$$\mathbb{E}\left[X_{t_2} \middle| \mathcal{F}_{t_1}\right] = X_{t_1} + \kappa(0)^2 \frac{(t_2 - t_1)^2}{9}.$$

No dynamic arbitrage implies $\kappa(0) = 0$.

- Conversely, if $\kappa(\nu) = k\,\nu$ with $k \geq 0$ then there is no dynamic arbitrage.

Optimal control of diffusion processes

Let $(X_t^u)_{t\in[0,T]}$ denote a controlled system with dynamics

$$dX_t^u = \mu\left(t, X_t^u, u_t\right)\, dt + \sigma\left(t, X_t^u, u_t\right) dW_t, \qquad X_0^u = X_0,$$

The agent has a **performance criterion** they wish to maximise

$$\boxed{\sup_{u\in\mathcal{A}} \mathbb{E}\left[G(X_T^u) + \int_0^T F(s, X_s^u, u_s)\, ds\right]}.$$

We define a class of problems indexed by time:

$$H(t,x) = \sup_{u \in \mathcal{A}_t} \mathbb{E}_{t,x}\left[G(X_T^u) + \int_t^T F(s, X_s^u, u_s)\, ds \right],$$

where $\left(X_s^{x,u}\right)_{s\in[t,T]}$ follows the dynamics

$$dX_s^u = \mu\left(t, X_s^u, u_s\right)\, ds + \sigma\left(s, X_s^u, u_s\right) dW_s, \qquad X_t^u = x,$$

Dynamic Programming Principle: The value function H satisfies

$$H(t,x) = \sup_{u \in \mathcal{A}_t} \mathbb{E}_{t,x}\left[H(T, X_T^u) + \int_t^T F(s, X_s^u, u_s)\, ds \right],$$

for all $t \in [0, T]$ and $x \in \mathbb{R}$.

The DPP connects the value function to its future expected value, regularised by the expected penalty F.

The infinitesimal version of the DPP gives the **Dynamic Programming Equation** (DPE) or the **Hamilton-Jacobi-Bellman** equation (HJB):

$$\partial_t H(t,x) + \sup_{u \in \mathcal{A}} (\mathcal{L}_t^u H(t,x) + F(t,x,u)) = 0$$

subject to the terminal condition $H(T,x) = G(x)$,

where $\mathcal{L}_t^u$ is the infinitesimal generator of the process $X_t^{x,u}$.

For the diffusion process

$$dX_t^u = \mu\left(t, X_t^u, u_t\right)\, dt + \sigma\left(t, X_t^u, u_t\right) dW_t, \qquad X_0^u = X_0,$$

the infinitesimal generator acts on functions H as follows:

$$\boxed{\mathcal{L}_t^u H(t,x) = \mu(t,x,u)\,\partial_x H(t,x) + \frac{1}{2}\sigma(t,x,u)^2\,\partial_{xx}^2 H(t,x)}$$

Sketch of the proof The dynamic programming principle gives

$$H(t,x) = \sup_u \left\{ \mathbb{E}\left[H(t+h, X^u_{t+h}) + \int_t^{t+h} F(s, X^u_s, u_s)\, ds \,\middle|\, X_t = x \right] \right\},$$

for any $h \in (0, T-t)$.

By Itô's formula we have

$$\begin{aligned} &H\left(t+h, X^u_{t+h}\right) - H(t,x) \\ &= \int_t^{t+h} (\partial_t + \mathcal{L}^u)\,[H](s, X^u_s) ds + \int_t^{t+h} \sigma(s, X^u_s, u_s) \partial_x H(s, X^u_s) dW_s\,, \end{aligned}$$

where $\mathcal{L}$ is the infinitesimal generator of X^u.

Divide by h on both sides and sending to 0, to obtain the HJB

$$\partial_t H(t,x) + \sup_u (\mathcal{L}^u_t H(t,x) + F(t,x,u)) = 0.$$

- An agent holds an **initial position** Q_0 at time $t = 0$ that they wish to **unwind** over a time window $[0, T]$.

- **Inventory**:

$$\boxed{dQ_t = \nu_t \, dt}$$

- **Price**:

$$\boxed{dS_t^{\nu} = \sigma \, dW_t + k \, \nu_t \, dt, \quad S_0 \in \mathbb{R}_+^{\star} \text{ is known}}$$

- **Cash**:

$$\boxed{dX_t = -\nu_t \, \tilde{S}_t \, dt = -\nu_t \, (S_t + \eta \, \nu_t) \, dt, \quad X_0 \in \mathbb{R} \text{ is known.}}$$

- **Admissible** strategies: **no unwind** constraint.

Performance criterion

- The agent's objective is to complete the liquidation, but we allow to fall short of this target (no fuel constraint) so $Q_T \neq 0$.
- We add a **terminal penalty** term to include any costs incurred when trading the terminal inventory Q_T.
- To encode risk aversion or urgency, the CJ framework includes a **running inventory penalty** that does not impact the wealth.

$$\mathbb{E}\left[\underbrace{X_T^\nu}_{\text{Terminal Cash}} + \underbrace{Q_T^\nu\,(S_T^\nu - \alpha Q_T^\nu)}_{\text{Terminal Execution}} - \underbrace{\phi\int_0^T (Q_u^\nu)^2}_{\text{Inventory Penalty}}\right]$$

$$= \mathbb{E}\left[\underbrace{X_T^\nu + Q_T^\nu\,S_T^\nu}_{\text{Terminal Wealth}} - \underbrace{(\alpha Q_T^\nu)^2}_{\text{Terminal Penalty}} - \underbrace{\phi\int_0^T (Q_u^\nu)^2}_{\text{Inventory Penalty}}\right],$$

Value function

$$H(t,x,S,q) = \sup_{\nu\in\mathcal{A}} \mathbb{E}_{t,x,S,q}\left[X_T^\nu + Q_T^\nu\, S_T^\nu - (\alpha Q_T^\nu)^2 - \phi\int_t^T (Q_u^\nu)^2\right],.$$

where $\mathbb{E}_{t,x,S,q}$ is the expectation conditional on $X_t^\nu = x$, $S_t^\nu = S$, and $Q_t^\nu = q$.

Writing the HJB

$$dQ_t = \nu_t\, dt, \qquad dS_t^{\nu} = \sigma\, dW_t + k\, \nu_t\, dt, \quad dX_t = -\nu_t\, (S_t + \eta\, \nu_t)\, dt$$

The HJB is

$$0 = \partial_t H + \sup_{\nu} \left\{ \nu\, \partial_q H + k\, \nu\, \partial_S H + \frac{1}{2} \sigma^2\, \partial_{SS} - \nu\, (S + \eta\, \nu)\, \partial_x H - \phi\, q^2 \right\}$$

with terminal condition

$$H(T, x, S, q) = x + S\, q - \alpha\, q^2, \quad \forall\, (x, S, q) \in \mathbb{R}^3.$$

Rearrange to obtain

$$0 = \left(\partial_t + \frac{1}{2}\sigma^2\,\partial_{SS}\right) H - \phi\, q^2 + \sup_{\nu}\left\{-\nu\,(S + \eta\,\nu)\,\partial_x H + k\,\nu\,\partial_S H + \nu\,\partial_q H\right\},$$

with terminal condition

$$H(T, x, S, q) = x + S\,q - \alpha\,q^2, \quad \forall\,(x, S, q) \in \mathbb{R}^3.$$

The first-order condition allows us to obtain the optimal trading speed as a function of the value function (feedback form):

$$\boxed{\nu^{\star} = \frac{1}{2\,\eta}\,\frac{-S\,\partial_x H + k\,\partial_S H + \partial_q H}{\partial_x H}.}$$

Substitute the optimal feedback control into the HJB to find

$$0 = \left(\partial_t + \frac{1}{2}\sigma^2\,\partial_{SS}\right) H - \phi\,q^2 + \frac{1}{4\,\eta}\,\frac{\left(-S\,\partial_x H + k\,\partial_S H + \partial_q H\right)^2}{\partial_x H},$$

with terminal condition

$$H(T, x, S, q) = x + S\,q - \alpha\,q^2.$$

We must propose an **ansatz**. The terminal condition suggests

$$\boxed{H(t,x,S,q) = \underbrace{x}_{\text{accumulated cash}} + \underbrace{q\,S}_{\text{value of shares}} + \underbrace{h(t,S,q)}_{\text{added value}},}$$

where $h(t,S,q)$ is still to be determined.

Use

$$\partial_x H = 1, \quad \partial_S H = q + \partial_S h, \quad \partial_q H = S + \partial_q h, \quad \partial_{SS} H = \partial_{SS} h$$

to simplify the HJB to

$$0 = \left(\partial_t + \frac{1}{2}\sigma^2\,\partial_{SS}\right) H - \phi\,q^2 + \frac{1}{4\,\eta}\,\frac{\left(-S\,\partial_x H + k\,\partial_S H + \partial_q H\right)^2}{\partial_x H},$$

$$\Longrightarrow 0 = \left(\partial_t + \frac{1}{2}\sigma^2\,\partial_{SS}\right) h - \phi\,q^2 + \frac{1}{4\,\eta}\,\frac{\left(k\,\left(q + \partial_S h\right) + \partial_q h\right)^2}{\partial_x H}$$

subject to the terminal condition

$$h(T,S,q) = -\alpha\,q^2.$$

The PDE contains no explicit dependence on the state variable S and the terminal condition is independent of S
$\implies$ h does not depend on S.

We write $h(t,S,q) = h(t,q)$ and $\partial_S h(t,S,q) = 0$

The PDE for h becomes

$$\boxed{0 = \partial_t h - \phi\, q^2 + \frac{1}{4\,\eta}\,\left(k\,q + \partial_q h\right)^2\,,}$$

and the optimal feedback control simplifies to

$$\nu^\star = \frac{1}{2\,\eta}\,\frac{-S\,\partial_x H + k\,\partial_S H + \partial_q H}{\partial_x H} \implies \boxed{\nu^\star = \frac{1}{2\,\eta}\,\left(k\,q + \partial_q h\right).}$$

The solution admits a separation of variables and takes the form of second-degree polynomial in q.

$$\boxed{h(t,q) = h_0(t) + h_1(t)\,q + h_2(t)\,q^2\,.}$$

Use

$$\begin{cases} \partial_t h = & \partial_t h_0(t) + h_1(t)\,q + h_2(t)\,q^2 \\ \partial_q h = & h_1(t) + 2\,h_2(t)\,q \end{cases}$$

to write

$$\begin{aligned} 0 &= \partial_t h - \phi\,q^2 + \frac{1}{4\,\eta}\,(k\,q + \partial_q h)^2 \\ \Longrightarrow 0 &= \left(\partial_t h_0(t) + h_1(t)\,q + h_2(t)\,q^2\right) - \phi\,q^2 + \frac{1}{4\,\eta}\,(k\,q + h_1(t) + 2\,h_2(t)\,q)^2 \\ \Longrightarrow 0 &= \left\{\partial_t h_2\,(t) - \phi + \frac{1}{4\,\eta}\,(k\ + 2\,h_2\,(t))^2\right\} q^2 \\ &\quad + \left\{\partial_t h_1\,(t) + \frac{1}{2\,\eta} h_1\,(t)\,(k\ + 2\,h_2\,(t))\right\} q + \left\{\partial_t h_0 - \frac{1}{4\,\eta}\,h_1\,(t)^2\right\} \end{aligned}$$

If the equality above is verified for all q, then

$$\begin{cases} 0 = & \partial_t h_2(t) - \phi + \frac{1}{4\eta}\,(k + 2\,h_2(t))^2 \\ 0 = & \partial_t h_1(t) + \frac{1}{2\eta} h_1(t)\,(k + 2\,h_2(t)) \\ 0 = & \partial_t h_0 - \frac{1}{4\eta}\,h_1(t)^2\ , \end{cases}$$

subject to the terminal conditions $h_2(T) = -\alpha,\ h_1(T) = 0,$ and $h_0(T) = 0.$

The solution to the ODE in h_1 is $h_1(t) = 0$.

The ODE in h_2 is a **Riccati** ODE. It can be solved explicitly:

$$\mathfrak{h}_2(t) = \sqrt{\eta\phi}\frac{1 + \zeta e^{2\gamma(T-t)}}{1 - \zeta e^{2\gamma(T-t)}}\ ,$$

where

$$\gamma = \sqrt{\frac{\phi}{\eta}} \quad \text{and} \quad \zeta = \frac{\alpha - \frac{1}{2}k + \sqrt{\eta\phi}}{\alpha - \frac{1}{2}k - \sqrt{\eta\phi}}.$$

We have fully determined the solution to the HJB.

The optimal trading strategy $\nu^\star$ can be obtained from the feedback form:

$$\nu_t^* = -\gamma \frac{\zeta \, e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta \, e^{\gamma(T-t)} - e^{-\gamma(T-t)}} Q_t^{\nu^*}.$$

And

$$Q_t^{\nu^*} = \frac{\zeta \, e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\zeta \, e^{\gamma T} - e^{-\gamma T}} Q_0.$$

Deterministic strategy. the optimal strategy is deterministic even though we did not restrict the space of admissible strategies.

Optimal trading curves and model parameters can be estimated prior to the trading window.

Fuel constraint. In the limit when the quadratic liquidation penalty goes to infinity, i.e., $\alpha \to \infty$,

$$\zeta \underset{\alpha\to\infty}{\longrightarrow} 1 \implies Q_t^{\nu^*} \underset{\alpha\to+\infty}{\longrightarrow} \frac{\sinh(\gamma(T-t))}{\sinh(\gamma T)} Q_0,$$

which does not depend on the permanent impact k.

Urgency of trading. As the running penalty increases, the optimal strategy aims to sell more assets sooner.

ϕ is the agent's urgency and penalises holding inventory.

When ϕ approaches zero (no aversion to risk), the optimal curve resembles a straight line

$$Q_t^{\nu^\star} \underset{\phi \to 0}{\longrightarrow} \frac{t}{T + k/\alpha}.$$

Fuel constraint. In the limit when the quadratic liquidation penalty goes to infinity, i.e., $\alpha \to \infty$,

$$\zeta \underset{\alpha \to \infty}{\longrightarrow} 1 \implies Q_t^{\nu^*} \underset{\alpha \to +\infty}{\longrightarrow} \frac{\sinh(\gamma(T-t))}{\sinh(\gamma T)} Q_0,$$

which **does not depend on the permanent impact k.**

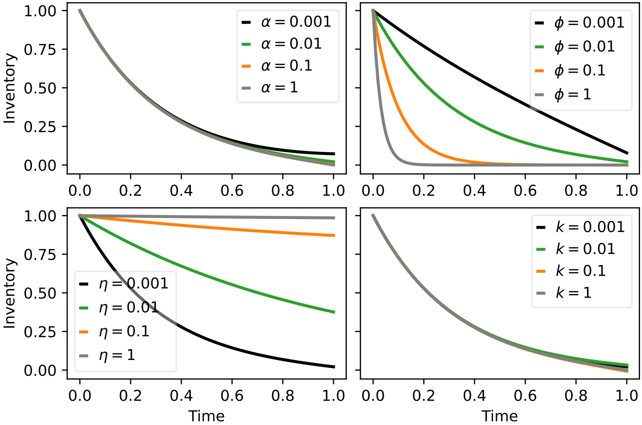


Figure: Optimal trading curves for multiple values of the parameters: terminal penalty α, running inventory penalty ϕ, execution costs η, and permanent impact k. The default parameter values are $\alpha = 0.01$, $\phi = 0.01$, $k = 0.001$, and $\eta = 0.001$. The other parameters are $T = 1$ and $Q_0 = 1$.

Gatheral, Jim (2010). "No-dynamic-arbitrage and market impact". In: **Quantitative finance** 10.7, pp. 749–759.

Consider the diffusion process X_t that solves the stochastic differential equation (SDE)

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t\,, \qquad X_0 = x\,, \tag{1}$$

where $\mu(t, x)$ and $\sigma(t, x)$ are functions of time and the state variable X_t.
theorem0NDefinition **Infinitesimal generator.** Let $g(t, x)$ be twice continuously differentiable. We define the infinitesimal generator $\mathcal{L}_t$ of the process X_t acting on g according to

$$\mathcal{L}_t g(t, x) = \frac{1}{2}\sigma^2(t, x)\, \partial_{xx}\, g(t, x) + \mu(t, x)\, \partial_x\, g(t, x)\,. \tag{2}$$

Almgren-Chriss in continuous-time Stochastic Optimal control Cartea-Jaimungal framework References Appendi
= Infinitesimal generators

One may also encounter the definition

$$\mathcal{L}_t g(t,x) = \lim_{t\to 0^+} \frac{\mathbb{E}[g(t,X_t)] - g(0,x)}{t}\,. \tag{3}$$

To see how we obtain expression (2) from (3) we proceed as follows. We use Ito's lemma to write

$$\begin{aligned} dg(t,X_t) &= \Big(\partial_t\, g(t,X_t) + \mu(t,X_t)\,\partial_x\, g(t,X_t) \\ &\quad +\frac{1}{2}\sigma^2(t,X_t)\,\partial_{xx}\, g\Big)\, dt + \sigma(t,X_t)\,\partial_x\, g(t,X_t)\, dW_t\,, \end{aligned}$$

which can be compactly written as

$$dg(t,X_t) = \left(\partial_t + \mathcal{L}_t\right)\, g(t,X_t)\, dt + \sigma(t,X_t)\,\partial_x\, g(t,X_t)\, dW_t\,.$$

Now define

$$M_t = g(t, X_t) - \int_0^t (\partial_t + \mathcal{L}_s)\, g(s, X_s)\, ds\,.$$

It is easy to show that M_t is a Martingale, thus

$$\mathbb{E}[g(t, X_t)] = g(0, x) + \mathbb{E}\left[\int_0^t (\partial_s + \mathcal{L}_s)\, g(s, X_s)\, ds\right]\,.$$

Now

$$\frac{d}{dt}\mathbb{E}[g(t,X_t)]\,|_{t=0} \quad = \quad \lim_{t\to 0^+} \frac{\mathbb{E}[g(t,X_t)] - g(t,x)}{t}$$
$$= \quad \lim_{t\to 0^+} \frac{1}{t}\,\mathbb{E}\left[\int_0^t (\partial_s + \mathcal{L}_s)\, g(s,X_s)\, ds\right] ,$$

and taking the limit inside the expectation operator (by the Lebesgue dominated convergence theorem) we show (2).

Infinitesimal generator of compensated Poisson process

Let N_t be a homogeneous Poisson process with intensity λ, and let $h(t)$ be a deterministic function of time. Moreover, let X_t satisfy the SDE

$$dX_t = h(t)\, dN_t - \lambda\, h(t)\, dt\,. \tag{4}$$

Ito's lemma for jumps:

$$\begin{aligned} df(t, X_t) &= \partial_t f(t, X_t) - \lambda\ h(t)\, \partial_x f(t, X_t) \\ &\quad + [f(t, X_t + h(t)) - h(t, X_t)]\ dN_t\,. \end{aligned}$$

and the infinitesimal operator is:

$$\mathcal{L}_t\, g(x) = \lambda\Big(\, [f(x + h(t)) - f(x)] - h(t)\, \partial_x\, g(x)\Big)\,. \tag{5}$$