

Market Microstructure and Algorithmic Trading

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- We consider the case of exponential decay and we write for $s \leq t$.

$$G(t - s) = \exp(-\beta(t - s))$$

The parameter ρ represents the rate of decay.

- We consider a linear instantaneous impact function

$$h : \nu \mapsto \lambda \nu.$$

The coefficient λ determines the distance between the midprices before and after an MO arrives in the LOB.

- The transient component of the midprice dynamics is now

$$\int_0^t \lambda \nu_s e^{-\beta(t-s)} ds.$$

- We define a new state variable I^ν , where $I_0 = 0$, that quantifies the accumulated transient impact:

$$I_t^\nu = \int_0^t \lambda \nu_s e^{-\beta(t-s)} ds \implies \boxed{dI_t^\nu = (\lambda \nu_t - \beta I_t^\nu) dt, \quad I_0 = 0.}$$

The dynamics of the problem are

$$\begin{cases} dQ_t^\nu &= \nu_t dt \\ dS_t^\nu &= \sigma dW_t + (\tilde{\lambda} \nu_t - \beta I_t^\nu) dt \\ dX_t^\nu &= - (S_t^\nu + \eta \nu_t) \nu_t dt \\ dI_t^\nu &= (\lambda \nu_t - \beta I_t^\nu) dt \end{cases}$$

The HJB is:

$$\begin{aligned} 0 = & \partial_t H - \phi q^2 + \frac{1}{2} \sigma^2 \partial_{SS} H - \beta I \partial_S H - \beta I \partial_I H \\ & + \sup_{\nu} \left\{ \tilde{\lambda} \nu \partial_S H + \lambda \nu \partial_I H + \nu \partial_q H - (S + \eta \nu) \nu \partial_x H \right\}, \end{aligned}$$

subject to the terminal condition

$$H(T, x, S, I, q) = x + qS - \alpha q^2.$$

The usual steps

1. **Ansatz** $H(t, x, S, l, q) = x + qS + h(t, q, l)$:

$$\implies 0 = \partial_t h - \phi q^2 - \beta l q - \beta l \partial_l h + \sup_u \left\{ \nu \left(\tilde{\lambda} q + \lambda \partial_l h + \partial_q h \right) - \eta \nu^2 \right\},$$

subject to the terminal condition

$$h(T, l, q) = -\alpha q^2.$$

2.

3. Linear and quadratic terms in q and $l \implies$

The ODEs in A, B, C are a matrix Riccati :

$$0 = P'(t) + Q + Y^T P(t) + P(t)Y + P(t)UP(t),$$

with terminal condition

$$P(T) = \begin{pmatrix} -\alpha & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$Q = \begin{pmatrix} -\phi + \frac{\tilde{\lambda}^2}{4\eta} & -\frac{\beta}{2} \\ -\frac{\beta}{2} & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} \frac{\tilde{\lambda}}{2\eta} & 0 \\ \frac{\tilde{\lambda}\lambda}{2\eta} & -\beta \end{pmatrix}, \quad \text{and } U = \frac{1}{\eta} \begin{pmatrix} 1 & \lambda \\ \lambda & \lambda^2 \end{pmatrix},$$

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- We mainly discussed optimal execution strategies that use market orders (MOs) only.
- Market orders **guarantee execution** but **incur costs** in the form of a bid-ask spread + they **adversely impact** the price.
- Limit orders are the main type of orders used by traders.
- LOs do **not guarantee** execution due to price and time priority. But they do not incur costs.

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Fill probability

- Let S_t denote the midprice. Assume an agent wishes to buy q share so they post a buy LO at the level $S_t - \delta^{\text{bid}}$.
- δ is the **depth** of the LO. It measures **price improvement**.

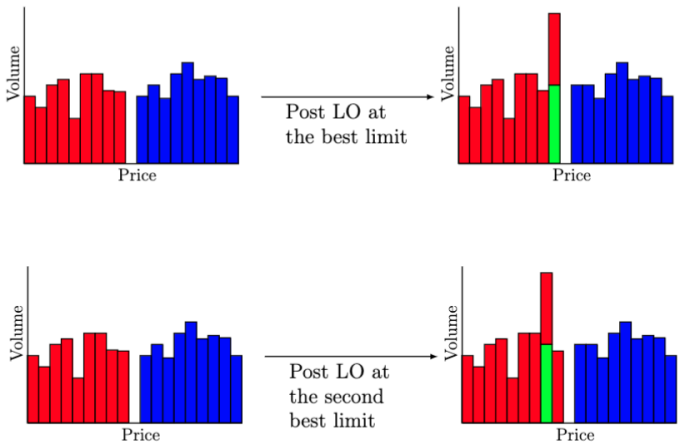
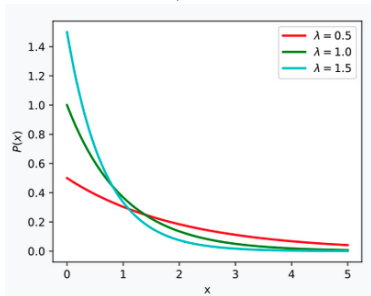


Figure 18: Limit order book.

- Let's figure out a law for the **fill probability**
- Let's assume that the volume of individual MOs is exponentially distributed with mean volume $\bar{v} = 1/\lambda$. The PDF:



- We assume the LOB, on average, is flat and block shaped, with fixed height A .
- For an LO posted at depth δ , the probability that an MO is large enough to fill it is:

$$\mathbb{P}[\text{execution of LO of depth } \delta] = \mathbb{P}[v > A\delta] = \exp\left\{-\frac{A}{\bar{v}}\delta\right\}.$$

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- An agent holds an **initial position** $Q_0 > 0$ at time $t = 0$ that they wish to **unwind** over a time window $[0, T]$.
- The agent controls the depth δ of LOs (of size 1) continuously sent at price level $S_t + \delta_t$.
- The agent continuously posts and cancels sell LOs:

At every instant in the trading window $[0, T]$, the agent reassesses market conditions and the inventory level, cancels any LO resting in the book, and posts a new sell LO at the optimal depth δ .

Dynamics

■ Price:

$$d S_t^\nu = \sigma d W_t$$

■ Inventory:

$$Q_t = Q_0 - N_t^\delta$$

■ Cash:

$$dX_t^\delta = (S_t + \delta_t) dN_t^\delta, \quad X_0 \in \mathbb{R} \text{ is known,}$$

The performance criterion

$$\mathbb{E} [X_T + Q_T^\nu (S_T^\nu - \alpha Q_T^\nu)] ,$$

where

$$\tau = T \wedge \min\{t : Q_t^\delta = 0\}$$

The value function

$$H(t, x, S, q) = \sup_{\delta \in \mathcal{A}} \mathbb{E}_{t,x,S,q} [X_\tau^\delta + Q_\tau^\delta (S_\tau - \alpha Q_\tau^\delta)] ,$$

where $\mathbb{E}_{t,x,S,q}$ is the expectation conditioned on (with a slight abuse of notation) $X_t = x$, $S_t = S$, and $Q_t = q$.

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Optimal control of diffusion processes

Let $(X_t^u)_{t \in [0, T]}$ denote a controlled system with dynamics

$$dX_t^u = \mu(t, X_t^u, u_t) dt + \sigma(t, X_t^u, u_t) dW_t + \gamma(t, X_t^u, u_t) dN_t^u, \quad X_0^u = X_0,$$

where N^u is a counting process with controlled stochastic intensity

$$\lambda_t^u = \lambda(t, X_t^u, u_t)$$

The agent has a **performance criterion** they wish to maximise

$$\sup_{u \in \mathcal{A}} \mathbb{E} \left[G(X_\tau^u) + \int_0^\tau F(s, X_s^u, u_s) ds \right].$$

where

$$\tau = \min\{T, \{t; X_t = \tilde{x}\}\}.$$

We define a class of problems indexed by time:

$$H(t, x) = \sup_{u \in \mathcal{A}_t} \mathbb{E}_{t,x} \left[G(X_T^u) + \int_t^T F(s, X_s^u, u_s) ds \right],$$

where $(X_s^{x,u})_{s \in [t, T]}$ follows the dynamics

$$dX_s^u = \mu(t, X_s^u, u_s) ds + \sigma(s, X_s^u, u_s) dW_s + \gamma(s, X_s^u, u_s) dN_s^u, \quad X_t^u = x,$$

Dynamic Programming Equation: The **Hamilton-Jacobi-Bellman** equation (HJB):

$$\partial_t H(t, x) + \sup_{u \in \mathcal{A}} (\mathcal{L}_t^u H(t, x) + F(t, x, u)) = 0$$

subject to the terminal conditions $H(T, x) = G(x)$, $H(t, \tilde{x}) = G(\tilde{x})$

where \mathcal{L}_t^u is the infinitesimal generator of the process $X_t^{x,u}$.

For the diffusion / jump process

$$dX_t^u = \mu(t, X_t^u, u_t) dt + \sigma(t, X_t^u, u_t) dW_t + \gamma(t, X_t^u, u_t) dN_t^u, \quad X_0^u = X_0,$$

the infinitesimal generator acts on functions H as follows:

$$\begin{aligned} \mathcal{L}_t^u H(t, x) &= \mu(t, x, u) \partial_x H(t, x) + \frac{1}{2} \sigma(t, x, u)^2 \partial_{xx}^2 H(t, x) \\ &\quad + \lambda(t, x, u) [H(t, x + \gamma(t, x, u)) - H(t, x)] \end{aligned}$$

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The solution

Our dynamics:

$$\begin{cases} dS_t^\nu &= \sigma dW_t \\ dQ_t &= -dN_t^\delta \\ dX_t^\delta &= (S_t + \delta_t) dN_t^\delta. \end{cases}$$

The value function solves the following dynamic programming equation (DPE):

$$\partial_t H + \frac{1}{2} \sigma^2 \partial_{SS} H \tag{5}$$

$$+ \sup_{\delta} \left\{ \lambda e^{-\kappa \delta} \underbrace{[H(t, x + (S + \delta), S, q - 1) - H(t, x, S, q)]}_{\text{change in the agent's value function when an MO fills the agent's LO}} \right\} = 0, \tag{6}$$

with boundary conditions

$$\begin{cases} H(t, x, S, 0) &= x, \\ H(T, x, S, Q) &= x + q(S - \alpha q). \end{cases}$$

The HJB is a non-linear partial integral differential equation (PIDE).

The solution

Next step?

Next step?

Use the ansatz

$$H(t, x, S, q) = x + qS + h(t, q).$$

Substitute into the PIDE and find that $h(t, q)$ satisfies

$$\begin{aligned} \partial_t h + \sup_{\delta} \{ \lambda e^{-\kappa\delta} [\delta + h(t, q-1) - h(t, q)] \} &= 0, \\ h(t, 0) &= 0, \\ h(T, q) &= -\alpha q^2. \end{aligned}$$

The solution

Next step ?

Next step ?

We can solve the sup term with a first order condition:

$$\begin{aligned} 0 &= \partial_{\delta} \{ \lambda e^{-\kappa\delta} [\delta + h(t, q - 1) - h(t, q)] \} \\ &= \lambda (-\kappa e^{-\kappa\delta} [\delta + h(t, q - 1) - h(t, q)] + e^{-\kappa\delta}) \\ &= \lambda e^{-\kappa\delta} (-\kappa [\delta + h(t, q - 1) - h(t, q)] + 1) , \end{aligned}$$

so the optimal depth δ^* in feedback form is given by

$$\delta^*(t, q) = \frac{1}{\kappa} + [h(t, q) - h(t, q - 1)].$$

The solution

Next step ?

Next step ?

We substitute δ in the DPE: Substitute the optimal depth in feedback form into the DPE to obtain the DPE for $h(t, q)$

$$\partial_t h + \frac{\tilde{\lambda}}{\kappa} \exp \{-\kappa [h(t, q) - h(t, q - 1)]\} = 0, \quad (7)$$

where $\tilde{\lambda} = \lambda e^{-1}$.

This is a coupled system of ODEs because q live on the grid $\{0, \dots, Q_0\}$.

The solution

Next step ?

Next step ?

We use the ansatz

$$h(t, q) = \frac{1}{\kappa} \log \omega(t, q),$$

so the new equation in ω is

$$\begin{aligned} 0 &= \partial_t h + \frac{\tilde{\lambda}}{\kappa} \exp \{ -\kappa [h(t, q) - h(t, q - 1)] \} \\ &= \frac{1}{\kappa} \frac{\partial_t \omega(t, q)}{\omega(t, q)} + \frac{\tilde{\lambda}}{\kappa} \frac{\omega(t, q - 1)}{\omega(t, q)}, \end{aligned}$$

which simplifies to

$$\partial_t \omega(t, q) + \tilde{\lambda} \omega(t, q - 1) = 0,$$

with terminal and boundary conditions

$$\omega(T, q) = e^{-\kappa \alpha q^2}, \quad \text{and} \quad \omega(t, 0) = 1.$$

The solution

The simple equation can be solved:

$$\omega(t, q) = \sum_{n=0}^q \frac{\tilde{\lambda}^n}{n!} e^{-\kappa \alpha (q-n)^2} (T-t)^n.$$

Use anzatz and feedback formula to find the optimal strategy.

$$\begin{cases} = \frac{1}{\kappa} \log \omega(t, q) \\ \delta^*(t, q) \end{cases} \quad \begin{aligned} & h(t, q) \\ & = \frac{1}{\kappa} + [h(t, q) - h(t, q-1)] \end{aligned}$$

$$\Rightarrow \delta^*(t, q) = \frac{1}{\kappa} \left[1 + \log \frac{\sum_{n=0}^q \tilde{\lambda}^n}{\sum_{n=0}^{q-1} \tilde{\lambda}^n} \right]$$

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Dependence of the quotes over the model parameters ?



Arroyo, Álvaro, Álvaro Cartea, Fernando Moreno-Pino, and Stefan Zohren (2023). “Deep Attentive Survival Analysis in Limit Order Books: Estimating Fill Probabilities with Convolutional-Transformers”. In: **Available at SSRN**.