Market Microstructure and Algorithmic Trading
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- We mainly discussed optimal execution strategies that use market orders (MOs) only.
- Market orders **guarantee execution** but **incur costs** in the form of a bid-ask spread + they **adversely impact** the price.
- Limit orders are the main type of orders used by traders.
- LOs do **not guarantee** execution due to price and time priority. But they do not incur costs.

- Let S_t denote the midprice. Assume an agent wishes to buy q share so they post a buy LO at the level $S_t - \delta^{\text{bid}}$.

- δ is the **depth** of the LO. It measures **price improvement**.

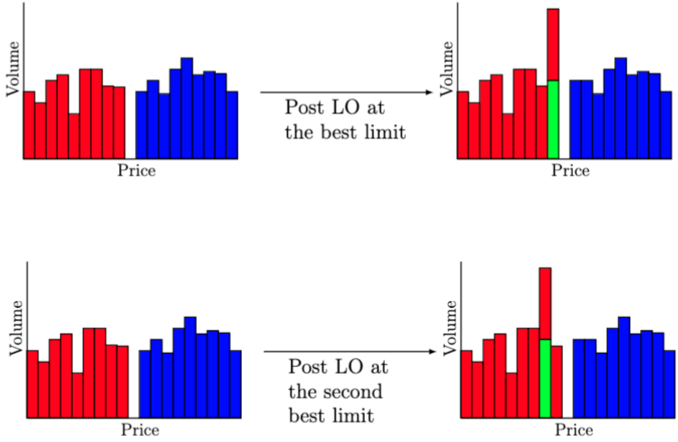


Figure 18: Limit order book.

- The larger the value of δ, the lower the probability that the order gets filled.
- The fill probability depends on the current state of the LOB, and the selling pressure.
- The deeper the LO is posted, the less likely it is that opposing MOs large enough to walk the LOB will arrive.

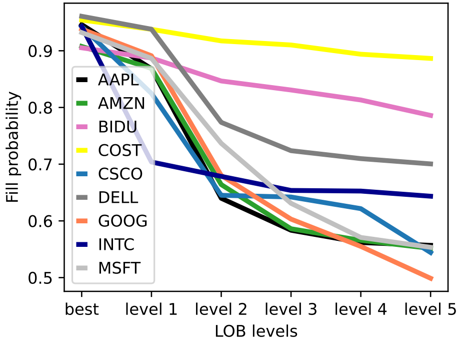
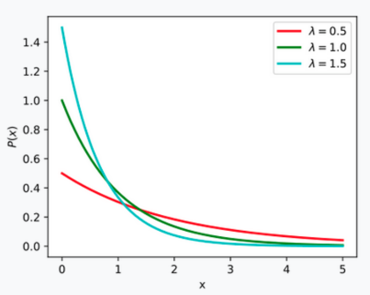


Figure: Fill probability at different levels of the LOB for multiple shares quoted on Nasdaq. Source: (Arroyo et al. 2023).

- Let's figure out a law for the **fill probability**

- Let's assume that the volume of individual MOs is exponentially distributed with mean volume $\overline{v} = 1/\lambda$. The PDF:



- We assume the LOB, on average, is flat and block shaped, with fixed height A.
- For an LO posted at depth δ, the probability that an MO is large enough to fill it is:

$$\boxed{\mathbb{P}[\text{execution of LO of depth } \delta] = \mathbb{P}[v > A\,\delta] = \exp\left\{-\frac{A}{\overline{v}}\,\delta\right\}}.$$

- An agent holds an **initial position** $Q_0 > 0$ at time $t = 0$ that they wish to **unwind** over a time window $[0, T]$.

- The agent controls the depth δ of LOs (of size 1) continuously sent at price level $S_t + \delta_t$.

- The agent continuously posts and cancels sell LOs:

  At every instant in the trading window $[0, T]$, the agent reassesses market conditions and the inventory level, cancels any LO resting in the book, and posts a new sell LO at the optimal depth δ.

- Market orders at the ask side arrive at a rate λ.
- The number of MOs that reached the agent's LOs at the depth δ_t is counted with a counting process $\left(N_t^\delta\right)_{t\in[0,T]}$.
- The counting process has intensity $\Lambda(\delta)$ that depends on δ and we write

$$\boxed{\Lambda(\delta) = \lambda\, \exp(-\kappa\,\delta)\,.} \tag{1}$$

- $\kappa > 0$ is the exponential decay parameter. It increases with available liquidity, and decreases with average trade size.

Dynamics

- **Price**:

$$\boxed{d\,S_t^{\nu} = \sigma\, d\,W_t}$$

- **Inventory**:

$$\boxed{Q_t = Q_0 - N_t^{\delta}}$$

- **Cash**:

$$\boxed{dX_t^{\delta} = (S_t + \delta_t)\ dN_t^{\delta}, \quad X_0 \in \mathbb{R} \text{ is known,}}$$

The performance criterion

$$\mathbb{E}\left[X_T + Q^{\nu}_T\ (S^{\nu}_T - \alpha\, Q^{\nu}_T)\right],$$

where

$$\tau = T \wedge \min\{t:\ Q^{\delta}_t = 0\}$$

The value function

$$H(t,x,S,q) = \sup_{\delta\in\mathcal{A}} \mathbb{E}_{t,x,S,q}\left[X^{\delta}_{\tau} + Q^{\delta}_{\tau}(S_{\tau} - \alpha Q^{\delta}_{\tau})\right],$$

where $\mathbb{E}_{t,x,S,q}$ is the expectation conditioned on (with a slight abuse of notation) $X_t = x$, $S_t = S$, and $Q_t = q$.

Optimal control of diffusion processes

Let $(X_t^u)_{t\in[0,T]}$ denote a controlled system with dynamics

$$dX_t^u = \mu\,(t, X_t^u, u_t)\; dt + \sigma\,(t, X_t^u, u_t)\, dW_t + \gamma\,(t, X_t^u, u_t)\, dN_t^u, \qquad X_0^u = X_0,$$

where N^u is a counting process with controlled stochastic intensity

$$\lambda_t^u = \lambda\,(t, X_t^u, u_t)$$

The agent has a **performance criterion** they wish to maximise

$$\boxed{\sup_{u\in\mathcal{A}} \mathbb{E}\left[G(X_\tau^u) + \int_0^\tau F(s, X_s^u, u_s)\, ds\right]}.$$

where

$$\tau = \min\{T, \{t;\; X_t = \tilde{x}\}\}.$$

We define a class of problems indexed by time:

$$H(t,x) = \sup_{u\in\mathcal{A}_t} \mathbb{E}_{t,x}\left[G(X^u_\tau) + \int_t^\tau F(s, X^u_s, u_s)\,ds\right],$$

where $\left(X_s^{x,u}\right)_{s\in[t,T]}$ follows the dynamics

$$dX^u_s = \mu\left(t, X^u_s, u_s\right)\,ds + \sigma\left(s, X^u_s, u_s\right)dW_s + \gamma\left(s, X^u_s, u_s\right)dN^u_s, \qquad X^u_t = x,$$

Dynamic Programming Equation: The **Hamilton-Jacobi-Bellman** equation (HJB):

$$\partial_t H(t,x) + \sup_{u\in\mathcal{A}}(\mathcal{L}^u_t H(t,x) + F(t,x,u)) = 0$$

subject to the terminal conditions $H(T,x) = G(x), H(t,\tilde{x}) = G(\tilde{x})$

where $\mathcal{L}^u_t$ is the infinitesimal generator of the process $X_t^{x,u}$.

For the diffusion / jump process

$$dX_t^u = \mu\,(t, X_t^u, u_t)\; dt + \sigma\,(t, X_t^u, u_t)\, dW_t + \gamma\,(t, X_t^u, u_t)\, dN_t^u, \qquad X_0^u = X_0,$$

the infinitesimal generator acts on functions H as follows:

$$\boxed{\begin{aligned} \mathcal{L}_t^u H(t,x) \quad = \quad & \mu(t,x,u)\,\partial_x H(t,x) + \tfrac{1}{2}\sigma(t,x,u)^2\,\partial_{xx}^2 H(t,x) \\ & +\lambda(t,x,u)\,[H(t,x+\gamma(t,x,u)) - H(t,x)] \end{aligned}}$$

Our dynamics:

$$\begin{cases} d\,S_t^\nu &= \sigma\,d\,W_t \\ dQ_t &= -dN_t^\delta \\ dX_t^\delta &= (S_t+\delta_t)\;dN_t^\delta. \end{cases}$$

The value function solves the following dynamic programming equation (DPE):

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H \tag{2}$$

$$+\sup_\delta \left\{ \lambda e^{-\kappa\delta} \underbrace{\left[H(t,x+(S+\delta),S,q-1) - H(t,x,S,q)\right]}_{\text{change in the agent's value function when an MO fills the agent's LO}} \right\} = 0, \tag{3}$$

with boundary conditions

$$\begin{cases} H(t,x,S,0) &= x, \\ H(T,x,S,Q) &= x + q\,(S-\alpha\,q)\,. \end{cases}$$

The HJB is a non-linear partial integral differential equation (PIDE).

Next step?

Next step?
Use the ansatz

$$\boxed{H(t,x,S,q) = x + q\,S + h(t,q)\,.}$$

Substitute into the PIDE and find that $h(t,q)$ satisfies

$$\begin{aligned}
\partial_t h + \sup_\delta \left\{ \lambda e^{-\kappa\delta} \left[\delta + h(t,q-1) - h(t,q) \right] \right\} &= 0, \\
h(t,0) &= 0, \\
h(T,q) &= -\alpha q^2.
\end{aligned}$$

Next step ?

Next step ?
We can solve the sup term with a first order condition:

$$\begin{aligned}
0 &= \partial_\delta \left\{ \lambda e^{-\kappa\delta} \left[\delta + h(t,q-1) - h(t,q) \right] \right\} \\
&= \lambda \left(-\kappa e^{-\kappa\delta} \left[\delta + h(t,q-1) - h(t,q) \right] + e^{-\kappa\delta} \right) \\
&= \lambda e^{-\kappa\delta} \left(-\kappa \left[\delta + h(t,q-1) - h(t,q) \right] + 1 \right),
\end{aligned}$$

so the optimal depth $\delta^\star$ in feedback form is given by

$$\boxed{\delta^*(t,q) = \frac{1}{\kappa} + [h(t,q) - h(t,q-1)].}$$

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Some comments:

- The term $1/\kappa$ optimises the instantaneous expected profits from sells one share.

- the expected price improvement from selling one share is $S+\delta$, minus the cost S:
$$(S+\delta-S)\,\Lambda(\delta) = \delta\, e^{-\kappa\,\delta}\,,$$
whose maximum is reached for $\delta = 1/\kappa$.

- the term $h(t,q) - h(t,q-1)$ can be interpreted as an additional wealth that the agent demands for changing their value function.

Next step ?

Next step ?

We substitute δ in the DPE: Substitute the optimal depth in feedback form into the DPE to obtain the DPE for $h(t, q)$

$$\boxed{\partial_t h + \frac{\tilde{\lambda}}{\kappa} \exp\left\{-\kappa\left[h(t, q) - h(t, q-1)\right]\right\} = 0\,,} \tag{4}$$

where $\tilde{\lambda} = \lambda\, e^{-1}$,

subject to boundary conditions $h(t, 0) = 0, \quad h(T, q) = -\alpha q^2$.

This is a coupled system of ODEs because q live on the grid $\{0, \ldots, Q_0\}$.

Next step ?

Next step ?

We use the ansatz

$$\boxed{h(t,q) = \frac{1}{\kappa}\log\omega(t,q)\,,}$$

so the new equation in ω is

$$\begin{aligned} 0 &= \partial_t h + \frac{\tilde{\lambda}}{\kappa}\exp\left\{-\kappa\left[h(t,q) - h(t,q-1)\right]\right\} \\ &= \frac{1}{\kappa}\frac{\partial_t\omega(t,q)}{\omega(t,q)} + \frac{\tilde{\lambda}}{\kappa}\frac{\omega(t,q-1)}{\omega(t,q)}, \end{aligned}$$

which simplifies to

$$\boxed{\partial_t\omega(t,q) + \tilde{\lambda}\,\omega(t,q-1) = 0\,,}$$

with terminal and boundary conditions

$$\omega(T,q) = e^{-\kappa\,\alpha\,q^2}\,, \quad \text{and} \quad \omega(t,0) = 1\,.$$

The simple equation can be solved:

$$\boxed{\omega(t,q) = \sum_{n=0}^{q} \frac{\tilde{\lambda}^n}{n!} e^{-\kappa\,\alpha (q-n)^2}\ (T-t)^n\,.}$$

Use anastzs and feedback formula to find the optimal strategy.

$$\begin{cases} h(t,q) & = \frac{1}{\kappa}\log\omega(t,q) \\ \delta^*(t,q) & = \frac{1}{\kappa} + [h(t,q) - h(t,q-1) \end{cases}$$

$$\Longrightarrow\ \boxed{\delta^*(t,q) = \frac{1}{\kappa}\left[1 + \log\frac{\sum_{n=0}^{q}\frac{\tilde{\lambda}^n}{n!}e^{-\kappa\alpha(q-n)^2}(T-t)^n}{\sum_{n=0}^{q-1}\frac{\tilde{\lambda}^n}{n!}e^{-\kappa\alpha(q-1-n)^2}(T-t)^n}\right]}\,,$$

Dependence of the quotes over time t and inventory q ?

Dependence of the quotes over time and inventory:

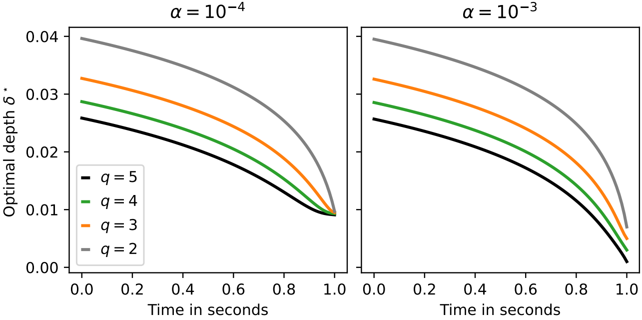


Figure: Optimal depth $\delta^\star$ of the agent's sell LOs as a function of time for different values of the inventory. The parameters are $\lambda = 50$, $T = 1$ minute, $\kappa = 100$, and $Q_0 = 5$.

Dependence of the quotes over the model parameters λ and κ ?

Dependence of the quotes over the model parameters:

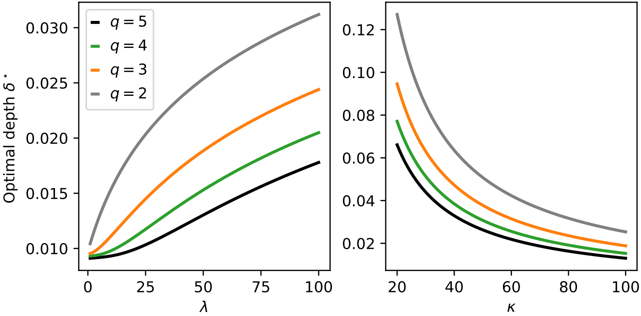


Figure: Optimal depth $\delta^\star$ of the agent's sell LOs as a function of time for different values of the inventory. The parameters of the left panel are $t = 0.8$ minutes, $T = 1$ minute, $\kappa = 100$, $\alpha = 10^{-4}$, and $Q_0 = 5$. The parameters of the right panel are $t = 0.8$ minutes, $T = 1$ minute, $\lambda = 50$, $\alpha = 10^{-4}$, and $Q_0 = 5$.

- OTC markets are off-exchange "quote-driven" markets that are based on a network of market makers that set prices at which liquidity takers can trade.
- To set prices, liquidity providers in OTC markets constantly stream bid and ask quotes.
- They also respond to requests made by their clients: Request For Quotes.
- Market makers in OTC markets compete for clients.

Let's find a model for competition among market makers in OTC markets:

- Assume a market maker observes the midprice S_t of an asset and streams bid and ask quotes $S_t - \delta_t^b$ and $S_t + \delta_t^a$.
- Assume λ^b is the **intensity of the selling** trading flow that a market maker would receive if the price of liquidity δ^b at the bid is zero.
- Similarly, λ^a is the **baseline intensity of the buying trading flow**.
- The buy/sell trading flow is sensitive to the price of liquidity $\delta^{b,a}$:

$$\boxed{\mathbb{P}[\text{execution of receiving an order at the bid/ask}] = \exp\left\{-\kappa^{b,a}\,\delta\right\}}.$$

- Intensity of order arrival at the bid/ask for the market maker:

$$\boxed{\lambda^{b,a}\exp\left\{-\kappa^{b,a}\,\delta^{b,a}\right\}}.$$

Setup

- A market maker (MM) operates in an OTC market for a trading period $[0, T]$ (usually one trading day).
- They are in charge of streaming bid/ask quotes for a single asset whose midprice is modelled by the process S.
- The agent controls the depths δ^b and δ^a of the bid and ask quotes $S_t - \delta_t^b$ and $S_t + \delta_t^a$ that they propose around the midprice S.
- At every instant in the trading window $[0, T]$, the MM reassesses market conditions and the inventory level and streams new quotes $S_t - \delta_t^b$ and $S_t + \delta_t^a$.

Order arrival

- The baseline intensity of order arrival at the bid side is λ^b and at the ask side is λ^a.
- Two counting (controlled) processes $N_t^{b,\delta}$ and $N_t^{a,\delta}$ model the number of sell orders and buy orders, respectively, that the market maker fills.
- The orders arrive at Poisson times with intensities $\Lambda^b(\delta^b)$ and $\Lambda^a(\delta^a)$:

$$\boxed{\begin{cases} \Lambda^b(\delta^b) = & \lambda^b \, e^{-\kappa^b \, \delta^b} \\ \Lambda^a(\delta^a) = & \lambda^a \, e^{-\kappa^a \, \delta^a} \, , \end{cases}} \tag{5}$$

- $\kappa^b, \kappa^a > 0$ are the rates of decay of the sell and buy pressure, respectively, as a function of the price of liquidity.

Inventory

- The MM manages inventory risk: they choose boundaries on how long or short their position is in the security.
- Let Q denote the MM's inventory. The MM chooses $\underline{q} < 0$ and $\overline{q} > 0$ such that they fill trades only when the inventory satisfies $Q \in (\underline{q}, \overline{q})$.
- We modify the intensity functions above and we set

$$\boxed{\begin{cases} \Lambda^b(\delta^b) = & \lambda^b \, e^{-\kappa^b \, \delta^b} \, 1_{Q<\overline{q}} \\ \Lambda^a(\delta^a) = & \lambda^a \, e^{-\kappa^a \, \delta^a} \, 1_{Q>\underline{q}} \, , \end{cases}}$$

- The inventory Q and the boundaries $\underline{q}, \overline{q}$ are integers that represent a number of shares.

Dynamics

- **Price**:

$$\boxed{dSt = \sigma\, dW_t\,, \quad S_0 \in \mathbb{R}_+ \text{ is known,}}$$

- **Inventory**:

$$\boxed{dQ_t^\delta = dN_t^{\delta,b} - dN_t^{\delta,a}, \quad Q_0^\delta = 0.}$$

- **Cash**:

$$\boxed{dX_t^\delta = -\left(S_t - \delta_t^b\right)\, dN_t^{\delta,b} + \left(S_t + \delta_t^b\right)\, dN_t^{\delta,a}, \quad X_0^\delta = X_0 \in \mathbb{R} \text{ know.}}$$

The performance criterion

$$H^{\delta}(t,x,S,q) = \mathbb{E}_{t,\boldsymbol{x},q,S}\left[X_T + Q_T^{\delta}(S_T^{\delta} - \alpha Q_T^{\delta}) - \phi \int_{\boldsymbol{t}}^{T} (Q_u)^2 du\right],$$

where $\alpha \geq 0$ is the terminal penalty from liquidating any remaining inventory, $\phi \geq 0$ scales the quadratic running inventory penalty.

The value function

$$H(t,x,S,q) = \sup_{\delta \in \mathcal{A}} H^{\delta}(t,x,S,q),$$

where $\mathbb{E}_{t,x,S,q}$ is the expectation conditioned on (with a slight abuse of notation) $X_t = x$, $S_t = S$, and $Q_t = q$.

The dynamics are

$$\begin{cases} dSt &= \sigma\, dW_t \\ dQ_t^\delta &= dN_t^{\delta,b} - dN_t^{\delta,a} \\ dX_t^\delta &= -\left(S_t - \delta_t^b\right)\, dN_t^{\delta,b} + \left(S_t + \delta_t^b\right)\, dN_t^{\delta,a} \end{cases}$$

The HJB is

$$0 = \partial_t H(t,x,q,S) + \frac{1}{2}\sigma^2\, \partial_{SS} H(t,x,q,S) - \phi\, q^2 \tag{6a}$$

$$+ \lambda^a \sup_{\delta^a} \left\{ e^{-\kappa^a\, \delta^a} \left(H(t, x + (S+\delta^a), q-1, S) - H(t,x,q,S)\right) \right\} 1_{q > \underline{q}} \tag{6b}$$

$$+ \lambda^b \sup_{\delta^b} \left\{ e^{-\kappa^b\, \delta^b} \left(H\left(t, x - (S-\delta^b), q+1, S\right) - H(t,x,q,S)\right) \right\} 1_{q < \bar{q}}, \tag{6c}$$

subject to the terminal condition

$$H(T,x,S,q) = x + q\, S - \alpha\, q^2.$$

Next step?

Next step? Propose the ansatz

$$\boxed{H(t,x,q,S) = x + q\,S + h(t,q).}$$

So

$$\begin{aligned}
H(t, x + (S+\delta^a), q-1, S) - H(t,x,q,S) =& x + (S+\delta^a) + (q-1)\ S + h(t,q-1) \\
& - x - q\,S - h(t,q) \\
=& \delta^a + h(t,q-1) - h(t,q) \\
H(t, x + (S-\delta^a), q+1, S) - H(t,x,q,S) =& \delta^b + h(t,q+1) - h(t,q)
\end{aligned}$$

Substitute in the HJB and find

$$0 = \partial_t h(t,q) - \phi q^2 + \lambda^a \sup_{\delta^a} \left\{ e^{-\kappa^a \delta^a} \left(\delta^a + h(t,q-1) - h(t,q)\right) \right\} 1_{q > \underline{q}} \tag{7a}$$

$$+ \lambda^b \sup_{\delta^b} \left\{ e^{-\kappa^b \delta^b} \left(\delta^b + h(t,q+1) - h(t,q)\right) \right\} 1_{q < \overline{q}}, \tag{7b}$$

Next step ?

Next step ? Solve the supremum to obtain the optimal feedback distances $\delta^{b,\star}$ and $\delta^{a,\star}$:

$$\delta^{b,*}(t,q) = \frac{1}{\kappa^b} - h(t,q+1) + h(t,q), \quad q \neq \overline{q}, \tag{8a}$$

$$\delta^{a,*}(t,q) = \frac{1}{\kappa^a} - h(t,q-1) + h(t,q), \quad q \neq \underline{q}. \tag{8b}$$

- The first components $1/\kappa^{a,b}$ optimise the instantaneous expected profit from a roundtrip trade: the expected profit from a roundtrip trade is

$$\delta^b \Lambda^b(\delta^b) + \delta^a \Lambda^a(\delta^a)$$

which is maximal for $\delta^b = 1/\kappa$ and $\delta^a = 1/\kappa$,

- The first component corresponds to the optimal strategy of a market maker who does not penalise inventory.

- The second component is related to changes in the value function due to an inventory change after trades are filled.

- **Assumption**: the intensity functions Λ^b and Λ^a decay at the same rate $\kappa = \kappa^a = \kappa^b$..

- We propose the following ansatz

$$\boxed{h(t,q) = \frac{1}{\kappa} \log \omega(t,q),}$$

- The HJB simplifies to

$$\begin{aligned} 0 =& \partial_t \omega(t,q) - \phi\, \kappa\, q^2\, \omega(t,q) \\ & + e^{-1} \lambda^a\, \omega\,(t, q-1)\, 1_{q > \underline{q}} + e^{-1} \lambda^b\, \omega\,(t, q+1)\, 1_{q < \overline{q}}, \end{aligned}$$

subject to the terminal condition

$$\omega(T,q) = \exp(-\kappa\, \alpha\, q^2).$$

The equation

$$
\begin{aligned}
0 =& \partial_t \omega(t,q) - \phi\,\kappa\, q^2\, \omega(t,q) \\
&+ e^{-1}\lambda^a\, \omega\,(t, q-1)\, 1_{q>\underline{q}} + e^{-1}\lambda^b\, \omega\,(t, q+1)\, 1_{q<\overline{q}},
\end{aligned}
$$

is a system of ODEs:

- The inventory q can only take the values $\{\underline{q}, \underline{q}+1, \dots, \overline{q}-1, \overline{q}\}$,
- For a fixed time t, $\omega(t,q)$ can only take the finitely many value $\{\omega(t,\underline{q}), \omega(t,\underline{q}+1), \dots, \omega(t,\overline{q}-1), \omega(t,\overline{q})\}$.
- Define the vector $\mathbf{w}(t) = \big(\omega(t,\underline{q}), \omega(t,\underline{q}+1), \dots, \omega(t,\overline{q}-1), \omega(t,\overline{q})\big)^{\intercal}$.
- **w** solves the ODE $\boxed{0 = \partial_t \mathbf{w}(t) + \mathbf{A}\,\omega(t),}$

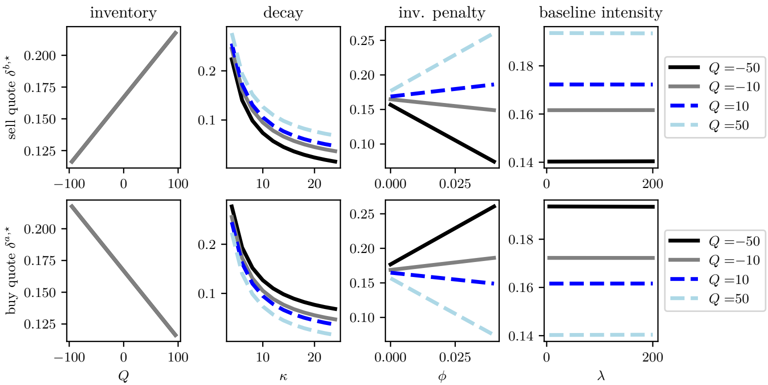
$$
\text{where } \mathbf{A} \in \mathcal{M}_{\overline{q}-\underline{q}+1}(\mathbb{R}) \text{ and } \mathbf{A}_{i,q} = \begin{cases} -\phi\,\kappa\, q^2, & i = q, \\ \lambda^a\, e^{-1}, & i = q-1, \\ \lambda^b\, e^{-1}, & i = q+1, \\ 0, & \text{otherwise,} \end{cases}
$$

The solution to the first-order homogeneous matrix ODE is straightforward:

$$\boxed{\mathbf{w}(t) = \exp\left(\mathbf{A}\left(T - t\right)\right)\mathbf{z},} \tag{9}$$

where the vector $\mathbf{z}$ is $(\overline{q} - \underline{q} + 1)-$dimensional that writes

$$\mathbf{z} = \left(e^{-\alpha\,\kappa\,\underline{q}^2}, \ldots, e^{-\alpha\,\kappa\,\overline{q}^2}\right)^{\mathsf{T}}.$$



Optimal distances $\delta^{b,\star}$ and $\delta^{a,\star}$ in as a function of model parameters for different values of the inventory. The default parameters are $\lambda = 50$, $T = 1$ minute, $\phi = 2 \times 10^{-3}$, $\kappa = 100$, and $Q_0 = 5$. The terminal time is $T = 30$ minutes and the terminal penalty is $\alpha = 10^{-4}$.

Arroyo, Álvaro, Álvaro Cartea, Fernando Moreno-Pino, and Stefan Zohren (2023). "Deep Attentive Survival Analysis in Limit Order Books: Estimating Fill Probabilities with Convolutional-Transformers". In: **Available at SSRN**.